Annotations to Abramowitz & Stegun

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The following is a compendium of additions and margin notes to the Handbook of Mathematical Functions by Abramowitz & Stegun (Dover 1972 edition), culled from personal annotations I have made to that reference over the years. I have found these formulas useful and handy to have around. Many are trivial restatements of what can already be found in the book, and a few are deeper, non-trivial relationships. Most of these are not 'mathematically significant', but are useful if one is just searching for an integral or some such: indeed, this is what it means to be a reference. They are put down here to be of some utility to the Internet community. It would be nice if future editions/revisions of the A&S reference were possible, and were to include such updates.

Sources & attribution: I derived all of these. I did not copy any of these from some other book/reference, except as noted. I've tripped over these while solving a large variety of other completely unrelated, but quite interesting problems.

These additional formulas are ordered according to the relevant chapter/paragraph of that book. Parenthetical comments justify the need for the inclusion of the formula, but are not meant to be added to the reference.

Without further ado:

3. Elementary Analytical Methods

3.6.8-a

$$\frac{1}{(1-x)^\alpha} = \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} x^k$$

(handy restatement of 3.6.8 in a non-obvious form)

3.7.12-a

$$\arg(x+iy) = \arctan(x/y)$$

(Just because arctan comes in a later chapter is no excuse to omit this very useful formula)

4. Elementary Transcendental Functions

4.1.5-a Discontinuity across the Branch Cut

$\ln(-x+i\varepsilon) - \ln(-x-i\varepsilon) = 2\pi i + O(\varepsilon)$ for real $x > 0$, and small, real ε.

(This follows obviously from 4.1.5 but is handy esp. for novice).

4.1.5-b

$$\ln(i\varepsilon) \quad - \quad \ln(-i\varepsilon) \quad = \quad \pi i \quad + \quad O(\varepsilon) \qquad \text{for} \qquad \text{small,} \qquad \text{real} \qquad \varepsilon.$$

(Non-intuitive statement about the limit on the imaginary axis).

4.7 Numerical Methods

A sequence of sines and cosines can be computed very rapidly (two multiplications, one addition each) and accurately with the following recursion relations: Let $s = \sin\Delta$ and $c = \cos\Delta$. Define $s_0 = \sin\theta$ and $c_0 = \cos\theta$, then $s_n = \sin(\theta + n\Delta)$ can be computed quickly, along with $c_n = \cos(\theta + n\Delta)$, by using $s_n = cs_{n-1} + sc_{n-1}$ and $c_n = cc_{n-1} - ss_{n-1}$. This method looses less than 3 decimals of floating point precision after 10 thousand iterations.

5.1 Exponential Integral

5.1.5-a

$$E_n(x) = x^{n-1}\frac{(-)^{n-1}}{(n-2)!}\left[E_1(x) - e^{-x}\sum_{m=0}^{n-2}\frac{(-)^m m!}{x^{m+1}}\right]$$

(This is related to 5.1.12 and 6.5.19 but is easier to work with than either; and is numerically more stable.)

5.1.23-a Special Values

$E_1(1) = 0.219383934$

(A handy-dandy value to have around)

5.1.25-a

Add Note: See also 5.1.49

5.1.51-a Asymptotic Expansion

The "hyperconvergent" can be obtained from the formal Euler Sum

$$\sum_{n=0}^{\infty} n! w^{n+1} = \int_{-\infty}^{0} \frac{e^{-x/w}}{1-x} dx$$

5.1.51-b

For values of negative n, see 6.5.32

6. Gamma Function

6.1.1-a

The following integrals look similar but are in fact very different:

$$\int_{0}^{\infty} \frac{t^{z-1}}{e^t + 1} dt \text{ See Riemann Zeta, section 23.2}$$

$$\int_{0}^{\infty} \frac{t^{z-1}}{e^t - 1} dt \text{ See Debye Function, section 27.1}$$

For integer z, see 6.4.1

6.1.1-b

$$\Gamma(1+\varepsilon) = \int_{0}^{\infty} e^{-x} x^{\varepsilon} dx = 1 + \varepsilon \int_{0}^{\infty} e^{-x} \ln x \, dx = 1 - \varepsilon \gamma \text{ For small, real } \varepsilon > 0.$$

(Mathematically "trivial", but handy if you just wanted to look up this integral).

6.1.21-a

$$\sum_{n=0}^{k} (-)^n \binom{k}{n} \binom{m+n+1}{n+1} = (-)^k \binom{m+1}{k+1} \Theta(m-k)$$

where $\Theta(x) = \left\{ \begin{array}{l} 0 \, for \, x < 0 \\ 1 \, for \, x \geq 0 \end{array} \right\}$ is the Heaviside step function.

6.3.21-a

$$\int_0^1 x^m \ln x \, dx = -1/(m+1)^2 \text{ for } m \neq -1$$

(Another handy integral deserving mention)

6.3.21-b

$$\int_0^1 (1-t)^z \ln t \, dt = \frac{\psi(1) - \psi(z+2)}{z+1}$$

and for integer $z = m$ we have

$$\int_0^1 (1-t)^m \ln t \, dt = -\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m+1}\right] \frac{1}{m+1}$$

(Another handy dandy integral to have around).

6.5.3-a

$$\Gamma(n,x) = \Gamma(n) e^{-x} \sum_{m=0}^{n-1} \frac{x^m}{m!} \text{ for n integer. See also 5.1.8}$$

(This is a special case that should be mentioned explicitly).

6.5.5-a

$$S_n(u) \equiv \int_0^\infty e^{-x} \left(1 + \frac{xu}{n}\right)^n dx = \sum_{k=0}^n \left(\frac{u}{n}\right)^k \frac{n!}{(n-k)!} = e^{n/u} \left(\frac{u}{n}\right)^n \Gamma\left(n+1, \frac{n}{u}\right)$$

(Occurs in certain types of stochastic equations; numerically unpleasant to evaluate.)

6.5.32

See also 5.1.51

6.8 Summation of Rational Series

(Section 6.8 should really be broken out into its own, and fortified with various utilitarian sums, e.g. the below. Sums occur in many problems, and should get a handy reference chapter, analogous to chapter 3, on their own.).

6.8.1

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+z)^2} = \frac{\pi^2}{\sin^2(\pi z)} \text{ See 4.3.92}$$

6.8.2

$$\sum_{n=0}^{\infty} \frac{1}{(n+z)^m} = \frac{(-)^m}{(m-1)!} \psi^{(m-1)}(z) \text{ See 6.4.10}$$

6.8.3

$$\sum_{n=1}^{\infty} \frac{1}{(n+z)(n+z+1)} = \frac{1}{1+z}$$

6.9 Formal Sums, Spectral Asymmetries

Some formally divergent sums can be given meaningful values through regularization. For example, $\lim_{t \to 0} \sum_{k=0}^{\infty} (-)^k (k+1) e^{-tk} = \frac{-1}{4}$ and thus we write, formally, $\sum_{k=0}^{\infty} (-)^k (k+1) = \frac{-1}{4}$ with the understanding that regulation took place. This is because other regulators, besides e^{-tk} can be used: for example, $e^{-t^2 k^2}$ provides excellent numerical stability, while $\frac{1}{k^s}$ in the limit $s \to 0$ is better suited to analytical treatments. General theories of series acceleration can be applied on formally divergent sums to get meaningful results.

6.9.1

$$\sum_{m=0}^{\infty} (-1)^m \binom{s-1}{m} = 0$$

6.9.2

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)} \binom{s-1}{m} = \frac{1}{s}$$

6.9.3

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)(m+2)} \binom{s-1}{m} = \frac{1}{s+1}$$

6.9.4

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)(m+2)(m+3)} \binom{s-1}{m} = \frac{1}{2(s+2)}$$

6.9.5

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)\ldots(m+p+1)} \binom{s-1}{m} = \frac{1}{p!(s+p)}$$

6.9.6

$$\sum_{m=0}^{\infty} (-1)^m m(m-1)\ldots(m-p) \binom{s-1}{m} = 0$$

6.9.7

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(m+2)} \binom{s-1}{m} = \frac{1}{s(s+1)}$$

Follows from above, & etc.

6.10 Finite Sums

(I copied these sums from some other book; they belong here.)

6.10.1

$$\sum_{k=1}^{n} k^4 = [n(n+1)(2n+1)(3n^2+3n+1)]/30$$

6.10.2

$$\sum_{k=1}^{n} k^5 = [n^2(n+1)^2(2n^2+2n-1)]/12$$

6.10.3

$$\sum_{k=1}^{n} k^6 = [n(n+1)(2n+1)(3n^4+6n^3-3n+1)]/42$$

6.10.4

$$\sum_{k=1}^{n} k^7 = [n^2(n+1)^2(3n^4 + 6n^3 - n^2 - 4n + 2)]/24$$

6.10.5

$$\sum_{k=1}^{n} (2k - 1) = n^2$$

6.10.6

$$\sum_{k=1}^{n} (2k - 1)^2 = n(4n^2 - 1)/3$$

6.10.7

$$\sum_{k=1}^{n} (2k - 1)^3 = n^2(2n^2 - 1)$$

6.10.8

$$\sum_{k=1}^{n} k(k + 1)^2 = [n(n+1)(n+2)(3n+5)]/12$$

6.11 Divergent Sums

Formally divergent sums that can be written as limits of convergent sums.

6.11.1

$$\lim_{t \to 0} \sum_{k=0}^{\infty} (-1)^k e^{-tk} = \frac{1}{2}$$

6.11.2

$$\lim_{t \to 0} \sum_{k=0}^{\infty} (-1)^k (k+2)e^{-tk} = \frac{3}{4}$$

6.11.3

$$\lim_{t \to 0} \sum_{k=0}^{\infty} (-1)^k (k+2)(k+3)e^{-tk} = \frac{7}{4}$$

6.11.4

$$\lim_{t \to 0} \sum_{k=0}^{\infty} (-1)^k (k+2)(k+3)(k+4)e^{-tk} = \frac{45}{8}$$

6.11.5

$$\lim_{t \to 0} \sum_{k=0}^{\infty} (-1)^k (k+2)(k+3)(k+4)(k+5)e^{-tk} = \frac{93}{4}$$

6.11.6

These are readily obtained[1] by considering the binomial generating function. That is, define

$$\begin{aligned} A_m(x) &= \sum_{k=0}^{\infty} \frac{\Gamma(k+m+2)}{\Gamma(k+2)} (-x)^k \\ &= -\frac{\Gamma(m+1)}{x} \sum_{k=1}^{\infty} \binom{k}{m} (-x)^k \\ &= \frac{\Gamma(m+1)}{x} \left(1 - \frac{1}{(1+x)^{m+1}}\right) \end{aligned}$$

and so the above sums are given by

$$A_m \equiv \lim_{x \to 1} A_m(x) = \Gamma(m+1) \left(\frac{2^{m+1}-1}{2^{m+1}}\right)$$

7. Error Function

7.1.4-a Integral Representations

$$erf\, z = 1 - \frac{2z}{\pi} e^{-z^2} \int_0^{\infty} \frac{e^{-t^2}}{t^2+z^2} dt \quad \text{See also 7.4.11}$$

7.2 Repeated Integrals

$i \neq \sqrt{-1}$, rather, i stands for integral.

(Using i to stand for 'integral' was a poor choice of notation for this entire section).

7.2.5 Repeated Integrals, Recurrence Relations

Let $I_n(z) = \int^z I_{n-1}(t) \, dt$ be the indefinite integral of erf, that is, $I_0(z) = \dfrac{2}{\sqrt{\pi}} \int^z e^{-t^2} \, dt$

then $I_n(z) = \dfrac{z}{n} I_{n-1}(z) + \dfrac{1}{2n} I_{n-2}(z) - \dfrac{z^{n-2}}{2n(n-2)!}$

(This looks like 7.2.5 but is the erf=1-erfc version of that relation. The entire section 7.2 should be redone with erf and erfc versions of the repeated integral.)

7.4-a Definite and Indefinite Integrals

$$\int_z^\infty [erfc(t)]^n \, e^{-t^2} \, dt = \frac{1}{n+1} \frac{\sqrt{\pi}}{2} [erfc(z)]^{n+1}$$

(Just another handy integral)

7.4-b

$$a \int_x^\infty e^{-a^2 z^2} erfc(bz) \, dz = \frac{\sqrt{\pi}}{2} erfc(ax) \, erfc(bx) - b \int_x^\infty erfc(az) \, e^{-b^2 z^2} \, dz$$

(Sadly, there's no closed form for this beastie).

7.4-c

$$\int_z^\infty \left(1 - 2a^2 t^2\right) e^{-a^2 t^2} erfc(bt) \, dt = \frac{1}{\sqrt{\pi}} \frac{b}{a^2 + b^2} e^{-\left(a^2 + b^2\right) z^2} - z e^{-a^2 z^2} erfc(bz)$$

7.4-d

Many of the integrals in section 7.4 can be obtained by writing $\int_0^\infty f(x) \, erf(zx) \, dx = \int_0^\infty dx \, f(x) \, x \int dz \, e^{-z^2 x^2}$ and then doing the x integral first.

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10. Bessel Functions of Fractional Order

10.1.4-a Asymptotic Expansions

For x real, $x \to \infty$,

$$j_n(x) = \frac{1}{x} \sin(x - n\pi/2) + O\left(\frac{1}{x^2}\right)$$

$$y_n(x) = \frac{-1}{x} \cos(x - n\pi/2) + O\left(\frac{1}{x^2}\right)$$

10.1.4-b

For fixed, real x and $n \to \infty$

$$j_n(nx) \to \frac{1}{(2n+1)\sqrt{2}} \left(\frac{ex}{2}\right)^n \quad \text{By use of Sterling's formula.}$$

10.1.10-a Asymptotic Expansions

$$f_n(z) = \frac{(-)^{n/2}}{z} + O\left(\frac{1}{z^3}\right) \quad \text{for } n \text{ even, } n \text{ positive or negative, and}$$

$$f_n(z) = \frac{n(n+1)}{2} \frac{(-)^{(n-1)/2}}{z^2} + O\left(\frac{1}{z^4}\right) \quad \text{for } n \text{ odd, } n \text{ positive or negative.}$$

10.1.10-b

Thus, for k even, $k \geq 0$ we have

$$j_k(z) = \frac{(-)^{k/2}}{z} \sin z + \frac{(-)^{k/2} k(k+1)}{z^2} \frac{}{2} \cos z + O\left(\frac{1}{z^3}\right)$$

and, for k odd, $k \geq 0$ we have

$$j_k(z) = \frac{(-)^{(k+1)/2}}{z} \cos z - \frac{(-)^{(k+1)/2} k(k+1)}{z^2} \frac{}{2} \sin z + O\left(\frac{1}{z^3}\right)$$

Although, see 10.1.4-a above for the correct treatment of the asymptotic phase angle.

(The phase angle is needed for quantum scattering problems).

11. Integrals of Bessel Functions

11.1 Simple Integrals of Bessel Functions

The $z \to \infty$ limit of these integrals is non-trivial. See 11.4.16, 11.4.17.

11.3.32-a

$$\int_0^z t J_{\nu-1}^2(t)\,dt = \frac{z^2}{2}\left[J_{\nu-1}^2(z) - J_\nu(z)J_{\nu-2}(z)\right] \text{ for } \mathscr{R}\nu > 0$$

(This closed form is easier to work with than the infinite sum given, and also reduces the order on the RHS.)

11.3.32-b

$$(2\nu-1)\int^z J_\nu(t)J_{\nu-1}(t)\frac{dt}{t} = z\left[J_\nu^2(z) - J_{\nu+1}(z)J_{\nu-1}(z)\right] + J_\nu(z)J_{\nu-1}(z)$$

11.3.34-a

Special case of 11.3.31.

11.3.35-a

See 9.1.76

11.3.35-b

$$\int_0^z J_\nu(t)J_{\nu+1}(t)\,dt = \sum_{n=0}^\infty J_{\nu+n+1}^2(z)$$

(Unlike 11.3.35, ν need not be integer in this formula)

11.3.36-a

Conjecture: Integrals of the type $\int t^n J_\xi(t) J_{\xi+m}(t)\,dt$ are solvable in closed form only for $n+m$ odd. (Disproof of this conjecture would bring an important addition to this subsection).

Integrals of the above form can be attacked using the recursion relations $J_{\nu-1}(z) = \frac{\nu}{z}J_\nu(z) + J_\nu'(z)$ and $J_{\nu+1}(z) = \frac{\nu}{z}J_\nu(z) - J_\nu'(z)$. (A useful set of integral recursion relations, suitable for numeric evaluation, are presented below.)

11.3.36-b

$$\int^z t J_v(t) J_{v+1}(t)\, dt = \frac{-2v}{2v-1} z J_v^2(z) + \frac{2v+1}{2v-1} \int^z t J_v(t) J_{v-1}(t)\, dt$$

11.3.36-c

$$\int^z t J_v(t) J_{v+1}(t)\, dt = \frac{-z}{2} J_v^2(z) + (2v+1) \int^z J_v^2(t)\, dt$$

11.3.36-d

$$\int^z t J_v(t) J_{v+1}(t)\, dt = -z J_v^2(z) + \int^z J_v^2(t)\, dt + \int^z t J_v(t) J_{v-1}(t)\, dt$$

11.3.36-e

$$\int^z J_v^2(t)\, dt = -J_v(z) J_{v-1}(z) + \int^z J_{v-1}^2(t)\, dt - \int^z J_v(t) J_{v-1}(t)\, \frac{dt}{t}$$

11.3.36-f

$$\int t J_v(t) J_{v+1}(t)\, dt = z^2 J_v(z) J_{v+1}(z) + \int^z t^2 J_{v+1}^2(t)\, dt - \int^z J_v^2(t)\, dt$$

11.3.41-a Integrals of Spherical Bessel Functions

(These occur in calculations of wave functions and are useful enough to deserve their own section).

The $j_n(z)$ are spherical Bessel functions, of chapter 10.

11.3.41-b

$$\int_0^\infty t^\mu j_v(t)\, dt = \sqrt{\pi}\, 2^{\mu-1} \frac{\Gamma\left(\dfrac{\mu+v+1}{2}\right)}{\Gamma\left(\dfrac{v-\mu+2}{2}\right)} \quad \text{for } \mathscr{R}(\mu+v) > -1 \text{ and } \mathscr{R}\mu < 1$$

11.3.41-c

$$\int_0^z t j_n(t)\left(j_{n-1}(t) - j_{n+1}(t)\right) dt = z j_n^2(z)$$

(follows from 11.3.33)

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11.3.41-d

$$\int_0^z t^2 \left[j_n^2(t) - j_{n+1}^2(t) \right] dt = z^2 j_n(z) j_{n+1}(z)$$

11.3.41-e

$$\int_0^z t^3 j_n(t) j_{n-1}(t) dt = \frac{z^3}{4} \left[(2n+1) j_n^2(z) - (2n-1) j_{n+1}(z) j_{n-1}(z) \right]$$

23. Bernoulli and Euler Polynomials - Riemann Zeta Function

23.1.3-a

$$B_n = \frac{-1}{(n+1)} \Sigma_{k=0}^{n-1} \left(\begin{array}{c} n+1 \\ k \end{array} \right) B_k$$

(Handy for generating large B_n numerically.)

23.2.5-a

γ_0 is Euler's constant, see 6.1.3. γ_i are called the Stieltjes constants. The first few are $\gamma_1 = -0.072815845$ and $\gamma_2 = -0.0096903$ and $\gamma_3 = 0.00205383$ and $\gamma_4 = 0.002325$.

23.3 Sums of Riemann Zeta Functions

(This is a new section, not in the current A&S. Turns out these are a special case of 6.4.9)

In the below, ν can be any complex value, not necessarily integer.

23.3.1

$$\zeta(\nu+2) = \sum_{k=0}^{\infty} \left(\begin{array}{c} k+\nu+1 \\ k \end{array} \right) [\zeta(k+\nu+2) - 1]$$

23.3.2

$$\sum_{k=0}^{\infty} \left(\begin{array}{c} k+\nu+1 \\ k+1 \end{array} \right) [\zeta(k+\nu+2) - 1] = 1 \text{ See also 6.1.33 for integer } \nu.$$

23.3.3

$$\sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} k+\nu+1 \\ k+1 \end{pmatrix} [\zeta(k+\nu+2)-1] = \frac{1}{2^{\nu+1}}$$

23.3.4

$$\sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} k+\nu+1 \\ k+2 \end{pmatrix} [\zeta(k+\nu+2)-1] = \nu[\zeta(\nu+1)-1] - \frac{1}{2^{\nu}}$$

23.3.5

$$\sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} k+\nu+1 \\ k \end{pmatrix} [\zeta(k+\nu+2)-1] = \zeta(\nu+2) - 1 - \frac{1}{2^{\nu+2}}$$

23.3.6

$$S_n \equiv \sum_{p=0}^{\infty} \begin{pmatrix} p+n \\ p \end{pmatrix} [\zeta(p+n+2)-1] = (-1)^n \left[1 + \sum_{k=1}^{n} (-1)^k \zeta(k+1) \right] \text{ For integer } n \geq$$
0.

Note that $S_0 = 1$ and $S_1 = \zeta(2) - 1$ and $S_2 = 1 - \zeta(2) + \zeta(3)$ and in general $S_n + S_{n+1} = \zeta(n+2)$, which is to be used in 23.3.2. Note $\lim_{n\to\infty} S_n = \frac{1}{2}$ which is numerically satisfied for n>20.

23.3.7

$$T_n \equiv \sum_{p=0}^{\infty} \begin{pmatrix} p+n-1 \\ p \end{pmatrix} [\zeta(p+n+2)-1] = (-1)^{n+1} \left[n+1 - \zeta(2) + \sum_{k=1}^{n-1} (-1)^k (n-k)\zeta(k+1) \right]$$
For integer $n \geq 1$. This follows from the observation $T_n + T_{n+1} = S_n$ when used in 23.3.6.

23.3.8

The above trick can be repeated to express $\sum_{p=0}^{\infty} \begin{pmatrix} p+n-k \\ p \end{pmatrix} [\zeta(p+n+2)-1]$ as a finite sum, for any integer k.

23.3.9

For integer $m > 0$,

$$\sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} k+\nu+1 \\ k \end{pmatrix} \zeta(k+\nu+2-m) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \zeta(\nu+2-j)$$

23.3.10

$$\sum_{k=0}^{\infty} (-1)^k (k+1) \zeta^2(k+2) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(nk+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \psi'(1+\frac{1}{n})$$

23.3.11

$$\sum_{k=0}^{\infty} (k+1) \zeta^2(k+2) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(nk-1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \psi'(1-\frac{1}{n})$$

Note this is a formal (divergent) sum that can be made meaningful through regularization. (XXX Need to do this).

References

[1] Stephen Crowley. Notes about the gkw operator. personal communication, 2010.