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BURST CHARACTERISTICS ASSOCIATED WITH THE SLOW ASSEMBLY OF FISSIONABLE MATERIALS

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ABSTRACT

This report is a sequel to LA-596, "Efficiency for Very Slow Assembly" by K. Fuchs. Herein are given estimates of temperature rise and pressure developed as a result of active material slowly being driven supercritical. Marginal assembly rates can thus be determined for both metal and solution assemblies above which the disassembly by thermal expansion is explosive in the sense that the active material or containing vessel is ruptured.

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BURST CHARACTERISTICS ASSOCIATED WITH THE SLOW ASSEMBLY OF FISSIONABLE MATERIALS

I. INTRODUCTION

The burst of fissions and associated energy release resulting from the very slow assembling of active material to a supercritical configuration has been treated by K. Fuchs in LA-596. In that report the multiplication rate α is considered to be composed of two parts $\alpha = \alpha_1 - \alpha_2$ where $\alpha_1(t)$ is the multiplication rate which would obtain if the temperature remained constant, the time dependence being associated with a mechanical change of configuration. When $\alpha_1(t)$ becomes greater than zero, the fission rate increases, the energy release leading to thermal expansion and consequent loss of reactivity. It is then assumed that $\alpha_2 = b\phi$ where b is a constant and ϕ is the number of fissions or energy release occurring after $\alpha_1(t)$ becomes greater than zero. This assumption together with the equation $d^2\phi/dt^2 = \alpha d\phi/dt$ permits a determination of $\phi(t)$. For future reference, the following brief summary of the results in LA-596 are given:

A). Instantaneous assembly to α_1.

$$\phi(t \to \infty) = 2\alpha_1/b$$

$$t_o = \frac{1}{\alpha_1} \log 2\alpha_1^2/b\dot{\phi}_o$$ where t_o is the time required for disassembly and $\dot{\phi}_o$ is the initial fission rate or power level.

B). Constant rate of assembly α_1 = at.

1) Total fissions or energy release per burst:

$$\phi = \frac{2\sqrt{a}}{b} \sqrt{2 \log 4a/b\dot{\phi}_o}$$

2) Time between bursts: $\tau_o \cong \frac{2}{\sqrt{a}}\sqrt{2 \log 4a/b\dot{\phi}_o}$

3) Width of burst: $\tau_1 \cong \frac{2}{\sqrt{a}}\sqrt{2/\log 4a/b\dot{\phi}_o}$

the reaction going thusly -- when the disassembly proceeds to $\alpha_2 = \alpha_1$, the fission rate or power level is a maximum and the system is driven subcritical to an extent indicated by $\alpha = -a\,\tau_o/2$, a time $\tau_o/2$ then being required to again bring the system to critical and another time $\tau_o/2$ required to develop fully the next burst.

As mentioned in LA-596, the assumption $\alpha_2 = b\phi$, b constant, means that the thermal expansion is at all times "adjusted" to the energy release. For fast (metal) systems, as the assembly rates are increased the inertia of the material becomes important, the thermal expansion lags the energy release and the size of bursts become larger than predicted by equation B-1. This enhancement of the burst is already of some importance for assembly rates slower than required for violent disassembly. On the other hand, for dilute solutions the obtainable values of α_1 are too small to permit the importance of inertia effects regardless of the assembly rates. Nevertheless, the disassembly may still be violent, and it is that range of assembly rates for which the first burst of energy is of the order required to give permanent disassembly that this report is concerned.

Toward this end, Section II gives the appropriate solutions for temperature and pressure for the case of a spherical assembly containing an energy source rising exponentially with time. In Section III, these solutions are utilized to determine the temperature rise and pressure developed as functions of assembly rate for 1) a typically metal system and 2) hypothetical liquid systems, a) having a completely free surface, and b) surrounded by a thin metal shell. The last Section considers the effect of partial containment for solution assemblies.

II. BASIC EQUATIONS AND SOLUTION FOR THE CASE OF AN ENERGY SOURCE INCREASING EXPONENTIALLY WITH TIME

The general hydrodynamical equations are:

Continuity equation $d\rho/dt + \rho \text{ div } V = 0$ II-1

Equation of motion $\rho dV/dt = - \text{div } \overrightarrow{\overrightarrow{P}}$ II-2

Energy conservation
$$\rho \frac{d}{dt} Q/\rho + \text{div } q = - P_{\alpha\beta} D_{\alpha\beta} + \rho \frac{d\phi}{dt} \qquad \text{II-3}$$

where ρ = density, V = velocity, $P_{\alpha\beta}$ = pressure components, $D_{\alpha\beta}$ = rate of strain components, Q = heat energy density = $\rho C_v T$ where C_v is the heat capacity per unit mass, and q = heat current density. ϕ is the energy source per unit mass which is here taken to have an exponential time dependence. For the equation of state, we take the linear form:

$$P_{ij} = \left[\left(\frac{\delta P}{\delta T} \right) T - (K - 2\mu/3)\nabla\cdot\xi \right] \delta_{ij} - 2\mu \Phi_{ij} \qquad \text{II-4}$$

where Φ_{ij} = strain components[(1)], K and μ = respectively, the modulus of volume elasticity and coefficient of shear, ξ = the displacement vector

$$\left(\frac{\delta\xi}{\delta t} = V \text{ and } \frac{\delta^2\xi}{\delta t^2} = dV/dt \right).$$

(1) Notation as used in Page - "Introduction to Theoretical Physics".

For the applications in mind, the times will be sufficiently short that the heat conduction plays no important part and this term can safely be dropped from equation II-3. Also the thermal expansions will be sufficiently small that the density terms appearing in equations II-2 and II-3 can be considered constant. The approximation that the energy conservation equation is equivalent to $C_v T = \phi$ will also be used in the final analysis, but this will be justified en route.

In the following, spherical symmetry is assumed in order to obtain the simplification of working with a single displacement variable ξ (the influence of geometrical symmetry being of importance for such cases as solutions which have partially open surfaces, the treatment of which is reserved for the last section). The preceding equations can then be rewritten as:

$$\frac{d\rho}{\rho} = -\left(\frac{\partial \xi}{\partial r} + \frac{2\xi}{r} \right) \qquad \text{II-5}$$

$$\rho \frac{\partial^2 \xi}{\partial t^2} = - \frac{\partial P_{rr}}{\partial r} - 2\,(P_{rr} - P_{\theta\theta})/r \qquad \text{II-6}$$

$$\rho \frac{\partial}{\partial t}\left[C_v T - \phi \right] = - P_{rr} \frac{\partial^2 \xi}{\partial t \partial r} - \frac{2P_{\theta\theta}}{r} \frac{\partial \xi}{\partial t} \qquad \text{II-7}$$

$$P_{rr} = \left(\frac{\partial P}{\partial T} \right) T - \left(K + \frac{4\mu}{3}\right)\left(\frac{\partial \xi}{\partial r} + \frac{2\xi}{r} \right) + 4\mu\xi/r \qquad \text{II-8}$$

$$P_{\theta\theta} = \left(\frac{\partial P}{\partial T} \right) T - \left(K - \frac{2\mu}{3}\right)\left(\frac{\partial \xi}{\partial r} + \frac{2\xi}{r} \right) - 2\mu\xi/r \qquad \text{II-9}$$

where $\delta\rho$ is the density change associated with the volume dilatation $(\partial\xi/\partial r + 2\xi/r)$.

<u>0th order approximation</u>: $C_v T = \phi(r,t) \sim e^{\alpha t} j_o(r/a)$.(2)

It is convenient to adopt the notation $\phi(r,t) = \phi\, j_o(r/a)$,

$T(r,t) = T\, j_o(r/a)$, etc., it then being understood that the coefficients ϕ and T have the characteristic $e^{\alpha t}$ time dependence.

After eliminating the pressure terms from II-6 by means of II-7, II-8, one obtains:

$$\rho \frac{\partial^2 \xi}{\partial t^2} - (K + 4\mu/3)\left(\frac{\partial^2 \xi}{\partial r^2} + \frac{2}{r}\frac{\partial \xi}{\partial r} - \frac{2}{r^2}\xi\right) = \frac{1}{C_v a}\frac{\partial P}{\partial T}\phi\, j_1(r/a) \qquad \text{II-10}$$

The forced solution is:

$\omega^2 = \frac{K + 4\mu/3}{\rho a^2}$

$$\xi(r,t) = \frac{1}{\rho C_v a(\alpha^2+\omega^2)}\frac{\partial P}{\partial T}\phi\, j_1(r/a) \equiv \xi_o j_1(r/a) \qquad \text{II-11}$$

(2) $j_o(r/a) = \sin(r/a) \div (r/a)$, this corresponding to the usual spatial distribution of fissions where $\pi a = r_e$, the extrapolated radius of the assembly. Throughout this report $j_k(x)$ will refer to the spherical Bessel function of order k (as given, for example, in Stratton's "Electromagnetic Theory").

while the free solution with a time dependence $e^{\alpha t}$ is:

$$\xi_1(r,t) = Ae^{\alpha t}\left[-ij_1(i\,\frac{\alpha}{\omega}\,r/a)\right] \qquad \text{II-12}$$

the coefficient A being determined such that the solution $\xi+\xi_1$ satisfies some specified boundary condition such as the vanishing of P_{rr} at the surface of the assembly. For the moment, we will not worry about ξ_1 which may be positive or negative and, in general, of a magnitude not substantially larger than ξ.

Another convenience is the adoption of the following units:

a) displacement ξ in units of a.

b) multiplication rate α in units of $\omega = V_o/a$

where V_o is the sound velocity $= \sqrt{(K + 4\mu/3)/\rho}$. The unit of time is thus $1/\omega$.

c) energy release per unit mass in units of V_o^2/γ where $\gamma = \frac{1}{\rho C_v}\frac{\delta P}{\delta T}$.

d) temperature rise in units of $V_o^2/\gamma C_v$.

e) pressure, P, in units of $K + 4\mu/3$.

The expression for displacement now becomes:

$$\xi_o = \frac{1}{1+\alpha^2}\,\phi \qquad \text{II-13}$$

while the temperature and pressure are given by:

$$T = \phi \qquad \text{II-14}$$

$$P_{rr}(r,t) = \frac{\alpha^2}{1+\alpha^2}\,\phi\, j_o(r/a) + \frac{4\mu}{K + 4\mu/3}\;\frac{\phi}{1+\alpha^2}\times \frac{a}{r}\, j_1(r/a) \qquad \text{II-15}$$

$$P_{\theta\theta}(r,t) = \frac{\alpha^2}{1+\alpha^2}\,\phi\, j_o(r/a) + \frac{2\mu}{K + 4\mu/3}\;\frac{\phi}{1+\alpha^2}\times \left\{ j_o(r/a) - \frac{a}{r}\, j_1(r/a) \right\} \qquad \text{II-16}$$

<u>1st order approximation</u>: $T(r,t) = \emptyset\, j_o(r/a) - \beta\, \emptyset^2 j_o^{\,2}(r/a)$

On substituting the 0th order solutions of pressure and displacement into the right hand side of the energy conservation equation, one obtains for the value of β:

$$\beta = \frac{\gamma}{2(1+\alpha^2)}\left[1 - \frac{1}{(1+\alpha^2)}\left\{\frac{K}{K + 4\mu/3} + \frac{4\mu/3}{K + 4\mu/3}\times \frac{j_2^{\,2}(r/a)}{j_o^{\,2}(r/a)}\right\}\right]$$

As the region of interest is for $\xi \ll 1$, then since γ is of the order one, it follows from equation II-13 and the expression for β that for $\alpha < 1$, $T = \emptyset\, j_o - \beta\, \emptyset^2 j_o^{\,2} \cong \emptyset\, j_o$, and the 0th order solutions for ξ, P, and T are satisfactory. For $\alpha > 1$, $\beta \cong \gamma/2(1 + \alpha^2)$, this value leading to the results:

$$\xi(r,t) = \frac{1}{1+\alpha^2}\,\phi\, j_1(r/a) - \frac{1}{2(1+\alpha^2)\,(1+2\alpha^2)}\,\gamma\,\phi^2 j_o(r/a) j_1(r/a) \qquad \text{II-13'}$$

$$T = \phi - \frac{1}{2(1 + \alpha^2)} \gamma \phi^2 \qquad \text{II-14}'$$

$$P_{rr} \cong P_{rr}(\text{0th order}) - \frac{1}{1 + 2\alpha^2} \gamma \phi^2 \qquad \text{II-15}'$$

$$P_{\theta\theta} \cong P_{\theta\theta}(\text{0th order}) - \frac{1}{1 + 2\alpha^2} \gamma \phi^2 \qquad \text{II-16}'$$

By replacing ϕ in the above equations with $(1 + \alpha^2)\xi$ as given in II-13, it can be seen that the 0th order equations give values of P, T, and ξ having fractional errors $\leq \xi$, i.e., small errors of the same magnitude as the error introduced by treating as a constant the density term appearing in the equation of motion. For this report, such errors are considered easily tolerable and only solutions based on the 0th order approximation $T = \phi$, that is, the equations II-13 to II-16, will be used.

III. APPLICATIONS TO SYSTEMS HAVING 1) TYPICALLY METAL CHARACTERISTICS, AND 2) TYPICALLY LIQUID CHARACTERISTICS

Preliminary remarks.

A) Qualitative features of the disassembly.

The energy release is governed by the equation $\ddot{\phi}/\dot{\phi} = \alpha = at - b\phi$. The rate of energy release is a maximum at the time t_o given by $at_o - b\phi = 0$, and the burst terminates very soon afterward. For the great percentage of the interval $0 \leq t \leq t_o$, the equation $\ddot{\phi}/\dot{\phi} = (at)$ holds and gives:

$$\dot{\phi} = \dot{\phi}_o \, e^{at^2/2} \quad \text{and} \quad \phi \simeq \dot{\phi}_o \, e^{at^2/2}/at$$

Thus at the time t_o given by $2a/b\dot{\phi}_o = e^{at_o^2/2}/(at_o^2/2)$ an amount of energy $\phi_1 \cong at_o/b$ has been released. For times just after t_o, α is negative and given by:

$$\alpha \cong at - b\left[\phi_1 + \dot{\phi}(t_o)\,(t - t_o)\right]$$

$$\cong -b\dot{\phi}_o \, e^{at_o^2/2} \, (t - t_o) = -a^2 t_o^2 (t - t_o) = \ddot{\phi}/\dot{\phi}$$

One then obtains: $\displaystyle\int_{t_o}^{\infty} d\phi \cong \sqrt{\frac{\pi}{2}} \, \frac{at_o}{b} \simeq at_o/b$

In summary, the energy release of the burst is:

$$\phi \cong 2at_o/b \qquad \text{III-1}$$

where the time to the "center of the burst" t_o, henceforth called burst duration, is given by:

$$2a/b\dot{\phi}_o = e^{at_o^2/2} / (at_o^2/2) \qquad \text{III-2}$$

(Note that with the approximation $\log 2a/b\dot{\phi}_o = at_o^2/2 - \log at_o^2/2 \cong \frac{at_o^2}{2}$, one obtains $t_o = \frac{1}{\sqrt{a}} \sqrt{2 \log 2a/b\dot{\phi}_o} \simeq \tau_o/2$ where τ_o is the time between bursts as given on Page 6. The validity of this approximation requires that $at_o^2/2 \gg 1$, a very mild condition which implies that the maximum fission rate is many times larger than the initial fission rate. This condition is assumed throughout the report.)

B) Regarding the parameters a and b.

If α_R is the prompt alpha for a delayed critical assembly and ΔM is the mass difference between delayed and prompt critical, then such a system made slightly supercritical has:

$$\alpha \cong \alpha_R \, (M - M_c)/\Delta M$$

where M is the actual mass and M_c is the prompt critical mass at existing conditions of temperature and pressure.

$$M_c = M_{co} + \frac{\partial M_c}{\partial \rho} \, \delta\rho = M_{co} - M_{co} \, n \, \overline{\delta\rho}/\rho$$

where M_{co} is the critical mass at the initial normal density, n is the density exponent (as in $M_c \sim \rho^{-n}$) and $\overline{\delta\rho}$ is a weighted average of the density change arising from the elevated temperatures. For a constant rate of assembly:

$$M = M_{co} + \frac{dM}{dt} t$$

and hence for this case,

$$\alpha \cong \frac{\alpha_R}{\Delta M} \left[\frac{dM}{dt} t - nM_{co} \left(- \frac{\overline{\delta\rho}}{\rho} \right) \right].$$

Thus,

$$a = \frac{\alpha_R}{\Delta M} \frac{dM}{dt} \qquad \text{III-3}$$

$$b = \frac{\alpha_R n M_{co}}{\Delta M} \left\{ - \frac{\overline{\delta\rho}}{\rho\phi} \right\} \qquad \text{III-4}$$

The term $-\delta\rho/\rho$ is related to the displacement ξ by means of the continuity equation II-5. For the special case when $\xi(r,t) = \xi_o j_1(r/a)$ then $-\delta\rho/\rho = \xi_o j_o(r/a)$. As an approximately correct weighting function for the averaging of the density change we will use $j_o(r/a)$ which is the spatial distribution of fissions. This then gives $-\frac{\overline{\delta\rho}}{\rho} = \xi_o \int j_o^2 dV / \int j_o dV \cong 3/4\, \xi_o$.

In general, ξ has the form $\xi_o j_1(r/a) + A\left[- i j_1(i\alpha r/a) \right]$ and $-\overline{\delta\rho}/\rho = C\xi_o$ where C is a constant dependent on the specific boundary condition. Since $\xi_o = \phi/(1 + \alpha^2)$,

$$-\frac{\overline{\delta\rho}}{\rho\phi} = \frac{C}{1 + \alpha^2} \text{ and } b = \frac{\alpha_R n M_{co}}{\Delta M} \frac{C}{(1 + \alpha^2)} \equiv b_o/(1 + \alpha^2) \qquad \text{III-5}$$

For constant rate of assembly, α is dependent on time but as far as b and the disassembly mechanism is concerned, its effective value is $\alpha \cong at_o$ as it is in this region that any lag of thermal expansion behind energy release is established. Equation III-2 now becomes:

$$\frac{2a}{b_o\dot{\phi}_o} = \frac{e^{at_o^2/2}}{(at_o^2/2)\left[1 + 2a(at_o^2/2)\right]} \qquad \text{III-6}$$

Equation III-6 is the basic equation for determining the burst duration, and thence the burst characteristics, in terms of the assembly rate a. The essential approximation involved is in the substitution of (at) for α in the displacement-energy release relation $\xi(t) = \phi(t)/(1 + \alpha^2)$ which was developed for constant α. A justification of this approximation in the range of interest can be given in the following manner. In the region $t < t_o$, the approximation $\xi/\phi = 1/\left[1 + (at)^2\right]$ is valid for $a/\omega^2 \ll 1$, i.e., $d\#/dt \ll \omega^2/\alpha_o$. As will be seen later, $\omega^2/\alpha_o \simeq \#8000\ sec^{-1}$ for the typical metal assembly or $\#6 \times 10^6\ sec^{-1}$ for the typical liquid assembly, and the range of interest is indeed for assembly rates of much smaller magnitudes. In the vicinity of $t = t_o$, the energy release function has the form $\Phi = \phi(t_o) + \dot{\phi}(t_o)\left[t - t_o\right] - C\left[t - t_o\right]^3/6$ this giving rise to a displacement of the form $\Xi = \phi(t_o) + \left\{\dot{\phi}(t_o) + C/\omega^2\right\}\left[t - t_o\right] - C\left[t - t_o\right]^3/6 + A\cos\omega(t - t_o) + B\sin\omega(t - t_o)$. At the time $t = t_o - 1/(at_o)$, the energy release is approximately $1/e^{th}$ of $\Phi(t_o)$ and hence the constants A, B, C, $\phi(t_o)$, and $\dot{\phi}(t_o)$ must be such that for $t \simeq t_o - 1/(at_o)$, $\Phi(t)$ matches $\phi(t) = \dot{\varphi}_o\, e^{at^2/2}/(at)$ and $\Xi(t)$ matches $\xi(t) = \dfrac{\phi(t)}{1 + (at)^2}$. The result of this matching gives $\Phi(t_o) \cong \frac{7}{3e} e^{at_o^2/2}/at_o \cong e^{at_o^2/2}/at_o$ and

$\Xi(t_o) = \phi(t_o) + A \cong \Phi(t_o)/\left[1 + (at_o)^2\right]$, which are the previously assumed relationships.

1) Typically metal assembly.

This first application will be to an untamped metal assembly and for the purpose of illustration the calculations will be given in more detail than for the remaining cases.

The displacement is:

$$\xi(r,t) = \xi_o j_1(r/a) + A\left[- i j_1(i\alpha r/a)\right]$$

where $\xi_o = \phi/(1 + \alpha^2)$ and A is chosen such that the radial pressure P_{rr} vanishes at the radius r_o of the metal sphere. Substituting the above expression for ξ in the equation of state gives:

$$P_{rr} = \frac{\alpha^2}{1 + \alpha^2}\phi\, j_o(r/a) + \frac{4\mu}{K + 4\mu/3}\,\frac{1}{1 + \alpha^2}\phi\,\frac{a}{r} j_1(r/a) -$$

$$A\alpha j_o(i\alpha r/a) + \frac{4\mu}{K + 4\mu/3} A \frac{a}{r}\left[- i j_1(i\alpha r/a)\right]$$

and the condition $P_{rr}(r = r_o) = 0$ leads to:

$$A = \frac{1}{\alpha}\;\frac{\phi}{1 + \alpha^2}\;(K + 4\mu/3)\frac{\left\{\alpha^2 j_o(r_o/a) + \frac{4\mu}{K+4\mu/3}\frac{a}{r_o} j_1(r_o/a)\right\}}{K j_o(i\alpha r_o/a) + \frac{4\mu}{3} j_2(i\alpha r_o/a)} \qquad \text{III-7}$$

and the equations for pressure and density change become:

$$P_{rr} = \frac{\phi}{1 + \alpha^2} \left\{ \alpha^2 \left[j_o(r/a) - F j_o(\frac{r_o}{a}) \right] + \frac{4\mu}{K + 4\mu/3} \times \left[\frac{a}{r} j_1(r/a) - F \frac{a}{r_o} j_1(r_o/a) \right] \right\} \qquad \text{III-8}$$

where $F = \left\{ j_o(i\alpha r/a) + \frac{4\mu}{3K} j_2(i\alpha r/a) \right\} / \left\{ j_o(i\alpha r_o/a) + \frac{4\mu}{3K} j_2(i\alpha \frac{r_o}{a}) \right\}$

$\cong$ 1 for $\alpha r_o/a << 1$

$$P_{\theta\theta} = \frac{\phi}{1 + \alpha^2} \left\{ \alpha^2 \left[j_o(r/a) - G\, j_o(r_o/a) \right] - \frac{2\mu}{K + 4\mu/3} \times \left[\frac{a}{r} j_1(r/a) - j_o(r/a) + \frac{2a}{r_o} G\, j_1(r_o/a) \right] \right\} \qquad \text{III-9}$$

where $G = \left\{ j_o(i\alpha r/a) - \frac{2\mu}{3K} j_2(i\alpha r/a) \right\} / \left\{ j_o(i\alpha r_o/a) + \frac{4\mu}{3K} j_2(i\alpha r_o/a) \right\}$

$\cong$ 1 for $\alpha r_o/a << 1$

$$- \frac{\overline{\delta\rho}}{\rho} = \overline{\xi_o j_o(r/a) + \alpha A\, j_o(i\alpha r/a)} \qquad \text{III-10}$$

$$\longrightarrow \frac{\phi}{1 + \alpha^2} \left[\overline{j_o(r/a) + \frac{4\mu}{3K} \frac{a}{r_o} j_1(r_o/a)} \right] \text{ for } \alpha << 1$$

$$\longrightarrow \frac{\phi}{1 + \alpha^2} \left[\overline{j_o(r/a)} \right] \qquad \text{for } \alpha >> 1$$

$\cong 1 \frac{\phi}{1+\alpha^2}$

It may be noted that the above limiting forms of $-\text{đ}\rho/\rho$ are characteristic in the sense that as α becomes larger, conditions imposed by the boundary become less important. The term $4\mu a j_1(r_o/a)/3Kr_o$ appearing in the $\alpha << 1$ form for $-\text{đ}\rho/\rho$ is the consequence of the particular boundary condition that $P_{rr} = 0$ at $r = r_o$. An indication of the effect which tamper surrounding the active metal sphere would have is obtained by changing a/r_o in this term. Thus, since $\pi a = r_e$ where r_e is the extrapolated radius for the fission distribution, reducing a/r_o artificially reduces the heating (energy release) near the metal surface, thereby making the metal in this region behave like tamper material. One concludes, as does LA-596, that for metal assemblies, the presence or absence of tamper makes little difference as far as the energy release associated with a given assembly rate is concerned.

Table I gives a list of properties which are intended to represent a typical untamped Uranium assembly. Table II gives a list of pressure, temperature, energy release, and burst duration as functions of assembly rate, this latter being expressed in dollars per second. The pressure components P_{rr} and $P_{\theta\theta}$ are evaluated at the center and radius of the assembly, respectively. If the assembly were to be ruptured during the burst, the value of $P_{\theta\theta}$ at the surface would have to be sufficiently negative to exceed the tensile strength of the material. As indicated by the values listed in Table II, the energy release and hence heating near the surface is apparently sufficient to prevent development of appreciable tensions. After the burst is over, momentum keeps the system expanding to an extent such that the tensions become of the same order as the listed values of

$P_{rr}(r = 0)$. In this manner, for assembly rates greater than \$100 sec^{-1}, the consequent tensions developed exceed the ~ 5000 atm. tensile strength of Uranium. Values of at_o are also given in Table II as these 1) indicate the important range of alpha, and 2) when divided by α_R give the peak supercriticality (in dollars) reached by means of the different assembly rates. Figure 1 gives a plot of peak pressure, temperature rise, and burst duration as functions of assembly rate.

TABLE I. Physical constants (approximate) intended to represent a typical untamped Uranium (235) assembly.

Pressure coefficient:	$\partial P/\partial T = 3.5 \times 10^{7}$ dynes cm^{-2} deg^{-1}
Modulus of volume elasticity:	$K = 10.2 \times 10^{11}$ dynes cm^{-2}
Coefficient of shear:	$\mu = 4.9 \times 10^{11}$ dynes cm^{-2}
Heat capacity per gram:	$C_v = .121 \times 10^{7}$ ergs deg^{-1} gm^{-1}
Density:	$\rho = 18.8$ gms cm^{-3}
Spatial distribution of fissions:	$j_o(r/a)$; $a = 3.5$ cm.
Assembly radius:	$r_o = 9$ cm.
Prompt alpha at delayed critical:	$\alpha_R = 1.0 \times 10^{6}$ sec^{-1}
Initial power level:	$\dot{\phi}_o = .02$ ergs/gm. sec.

Derived Constants

$$\gamma = \frac{1}{\rho C_v}\frac{\partial P}{\partial T} = 1.54$$

$$V_o = \sqrt{(K + \tfrac{4\mu}{3})/\rho} = 3.16 \times 10^{5} \text{ cm sec}^{-1}$$

$$\omega = V_o/a = .90 \times 10^{5} \text{ sec}^{-1}$$

Energy unit: $V_o^2/\gamma = 6.5 \times 10^{10}$ ergs gm^{-1}

Pressure unit: $K + 4\mu/3 = 18.7 \times 10^{11}$ dynes cm^{-2} $\simeq$ 1.85×10^{6} atm.

Temperature unit: $V_o^2/\gamma C_v = 5.35 \times 10^{4}$ °C.

TABLE II. Burst characteristics - Untamped metal assembly.

Assembly rate (dollars/sec)	at_o-in units of ω	$\phi(r=0)$ (joules/gm)	T(0)-in deg. C.	$P_{rr}(0)$ (atmospheres)	$P_{\theta\theta}(r_o)$ (atmospheres)	Burst duration t_o(seconds)
1	.083	1.2	10	75	- 55	.0075
4	.171	2.6	21	170	-110	.00385
10	.278	4.4	36	330	-172	.00250
40	.57	11.2	92	1300	-310	.00128
100	.91	25	206	4100	-320	.00082
400	1.87	125	1020	32000	1800	.00042

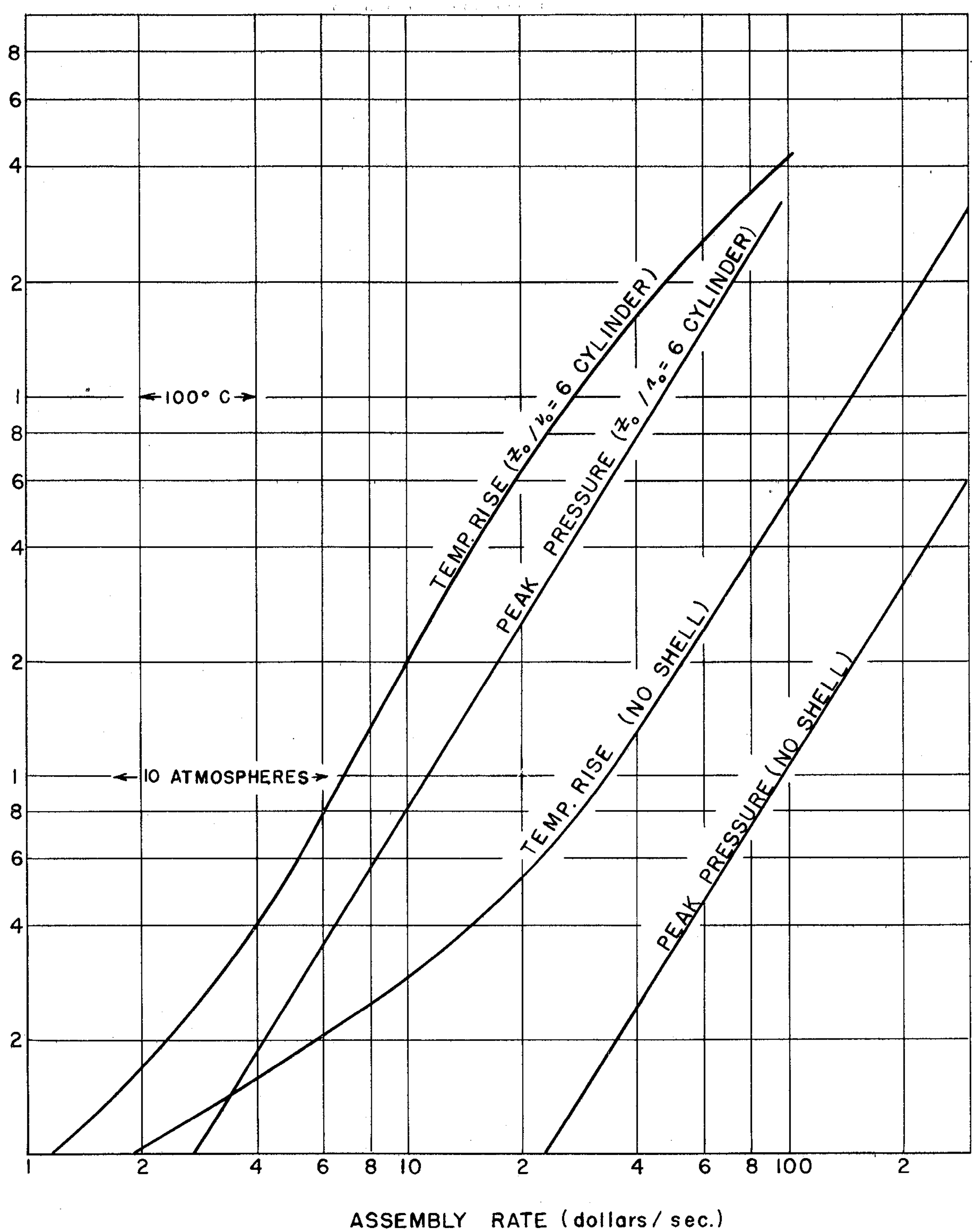

FIG. 1. Effect of partial containment in solution assemblies. A comparison of the $z_o/r_o = 6$ cylinder having a rigid wall and open faces with the hypothetical free surface assembly designated "no shell".

2) Liquid assembly surrounded by a thin metal shell. In the liquid the displacement is of the form:

$$\xi_{\ell} = \xi_o \left[j_1(r/a) + A_{\ell} \left\{ - i j_1(i\alpha r/a) \right\} \right]$$

while in the shell it is:

$$\xi_s = \xi_o \left[A_s \left\{ - i j_1(i\alpha r/L) \right\} + B_s \left\{ - i n_1(i\alpha r/L) \right\} \right]$$

where n_1 is the spherical Neumann function of order one and L is a length defined by $L = v_s/\omega = \sqrt{(K_s + 4\mu_s/3)/\rho_s}/\omega$

The three constants A_1, A_s, and B_s are determined from the three conditions a) vanishing of P_{rr} at the outside shell surface, b) and c) continuity of ξ and P_{rr} at the liquid-shell interface. For $\alpha << 1$ and small shell thickness, Δ , one obtains:

$$A_{\ell} \cong \frac{\dfrac{-3\Delta K_s a}{r_o K_L L\alpha} j_1(r_o/a) + \dfrac{\alpha r_o}{3L}\left[1 + \dfrac{3K_s}{4\mu_s}\right] j_o(r_o/a)}{\dfrac{r_o}{3L}\left[1 + \dfrac{3K_s}{4\mu_s} + \dfrac{3\Delta K_s}{r_o K_L}\right]} \qquad \text{III-11}$$

$$A_s \cong \frac{r_o L\alpha K_L}{3\Delta a K_s}\left[j_o(r_o/a) - \frac{1}{\alpha} A_{\ell}\right] \qquad \text{III-12}$$

$$B_s \cong -\frac{K_s}{4\mu_s}\left(\frac{\alpha r_o}{L}\right)^3 A_s \qquad \text{III-13}$$

where r_o is the radius of the liquid shell interface. In the expression for A_ℓ above, it is seen that for $\Delta \to \infty$, $A_\ell = -3aj_1(r_o/a)/r_o\alpha$, a value which corresponds to the case of a rigid container, $\xi(r_o) = 0$, and, because for most metals $K_S/K_L >> 1$ also corresponds very closely to the case of the thick metal shell.

The relation between density change and $\xi_o = \phi/(1 + \alpha^2)$ is:

$$- \overline{d\rho/\rho} = \xi_o \left[\overline{j_o(r/a)} + \alpha A_\ell \overline{j_o(i\alpha r/a)} \right] \cong \xi_o \left[\overline{j}_o + \alpha A_\ell \right] \qquad \text{III-14}$$

The pressure in the liquid is given by:

$$P_\ell \cong \xi_o \left[\alpha^2 j_o(r/a) - \alpha A_\ell \right] \qquad \text{III-15}$$

For shell thicknesses of the order of a couple millimeters or more and nominal assembly rates, the pressure developed is essentially determined by the second term in III-15 and is thus nearly constant throughout the liquid. The tangential tension developed in the shell is $-P_{\theta\theta} \cong r_o P_\ell / 2\Delta$.

Effect of gas evolution: The fact that most of the fission energy source goes into thermal energy makes it possible to incorporate gas evolution into the equation of state by means of a simple change in the pressure coefficient.

Thus, letting

V_t = volume per unit mass of solution

V_ℓ = volume of liquid per unit mass of solution

V_g = volume of gas per unit mass of solution

ϕ_ℓ = energy release required to produce one mole of gas = 3.1×10^{13} ergs

(based on .289 liters of hydrogen generated per KW minute operation of the hypo water boiler[3]).

[3] This datum on gas production was obtained from L.D.P. King.

then, since $V_t = V_\ell + V_g$, one has

$$V_t(\Delta P, \Delta T) = V_\ell(\Delta P, \Delta T) + \frac{RT}{P} \frac{C_v \Delta T}{\phi_1} \qquad \text{III-16}$$

where ΔP and ΔT are the changes in pressure and temperature associated with the energy release $\phi = C_v \Delta T$. T and P are the absolute temperature and pressure and R is the gas constant.

Writing V_ℓ as $V_\ell = V_o + (\frac{\partial V_\ell}{\partial T}) \Delta T + (\frac{\partial V_\ell}{\partial P}) \Delta P$ and $V_t - V_o = \Delta V_t$, equation III-16 becomes:

$$\Delta P = \frac{\left[\frac{\partial V_\ell}{\partial T} + \frac{RT}{P} \frac{C_v}{\phi_1}\right] \Delta T}{\left(-\frac{\partial V_\ell}{\partial P}\right)} - \frac{\Delta V_t}{\left(-\frac{\partial V_\ell}{\partial P}\right)}$$

$$\cong \left[\left(\frac{\partial P}{\partial T}\right)_{V_\ell} + K \frac{RT}{PV_t} \frac{C_v}{\phi_1}\right] \Delta T - K \frac{\Delta V_t}{V_t} \qquad \text{III-17}$$

where $K = -V_\ell \left(\frac{\partial P}{\partial V_\ell}\right)$, and $(\partial P/\partial T)_{V_\ell}$ are the bulk modulus and pressure coefficient of the liquid alone. The evolution of gas is thus taken into account by substituting

$$\left[\left(\frac{\partial P}{\partial T}\right)_{V_\ell} + K \frac{RT}{PV_t} \frac{C_v}{\phi_1}\right] \quad \text{for} \quad \frac{\partial P}{\partial T}$$

in the already developed relations, the effective values of P and T in the bracket term corresponding closely to the peak pressure and temperature developed in the burst; these effective values can be determined by successive guesses.

That the coefficient of $\Delta V_t/V_t$ is unchanged by gas evolution does not imply an unchanged compressibility, this being given by $\frac{1}{V_t} \frac{\delta \Delta V_t}{\delta \Delta P} \neq -1/K$.

Table III gives a list of the physical properties intended to characterize a typical liquid assembly. Table IV contains the estimates of peak pressure, etc., at various assembly rates for the three cases, 1) no shell, 2) 1/4 cm shell, and 3) rigid container. The elastic properties of the shell have been taken equal to those listed in Table I. Figure 2 gives a graph of temperature rise and peak pressure for the first two cases above. For the case of no shell where the pressures developed are low, the gas evolution plays an extremely important part in reducing the reactivity and limiting the energy release, the bracket term $\left[\left(\frac{\delta P}{\delta T}\right)_{V_\ell} + K \frac{RT}{PV_t} \frac{C_v}{\phi_1} \right]$ being approximately 70 times greater than $\left(\frac{\delta P}{\delta T}\right)_{V_\ell}$. With the solution contained by a metal shell or in general when the pressures developed are 70 atmospheres or more, the effect of gas evolution on burst size becomes of minor importance.

TABLE III. Physical constants (approximate) intended to represent a typical solution assembly.

Pressure coefficient for the liquid:	$\left.\frac{\delta P}{\delta T}\right\rvert_{V_\ell} = 1.9 \times 10^7$ dynes $cm^{-2} deg^{-1}$
Effective pressure coefficient for the solution: (to include effect of gas evolution)	$\frac{\delta P}{\delta T} = \left\{ 1.9 \times 10^7 + \frac{RT}{PV_o} \frac{C_v}{\phi_1} \right\}$
Modulus of volume elasticity:	$K = 4.2 \times 10^{10}$ dynes/cm^2
Heat capacity per gram:	$C_v = 4.19 \times 10^7$ ergs/gm.deg.
Density:	$\rho = 1$
Spatial distribution of fissions:	$j_o(r/a)$, $a = 8$ cm.
Assembly radius:	$r_o = \pi a$
Prompt alpha at delayed critical:	$\alpha_R = 100\ sec^{-1}$
Initial power level:	$\phi_o = .02$ ergs/gm.

Derived Constants

$$V = \sqrt{K/\rho} = 2.05 \times 10^5 \text{ cm. sec}^{-1}$$

$$\omega = V/a = 2.56 \times 10^4 \text{ sec}^{-1}$$

TABLE IV. Burst characteristics - Liquid assembly.

	Assembly rate (dollars/sec)	at_o in units of ω	Ø joules	Temp. ^oC.	Press. (atmos-pheres)	Burst duration (seconds)
1) no shell	.256	.00135	1.5	0.4	.0008	1.35
	2.56	.0045	4.9	1.2	.03	.45
	25.6	.015	30	7	1.1	.15
	76.8	.027	150	35	6.7	.090
	256	.051	990	240	46	.051
2) 1/4 cm metal shell	.256	.00146	70	17	80	1.46
	2.56	.0048	350	80	260	.48
	25.6	.016	1300	310	870	.16
3) rigid container	.256	.00150	320	75	770	1.50
	2.56	.0049	1100	260	2500	.49
	25.6	.0162	3700	880	8300	.16

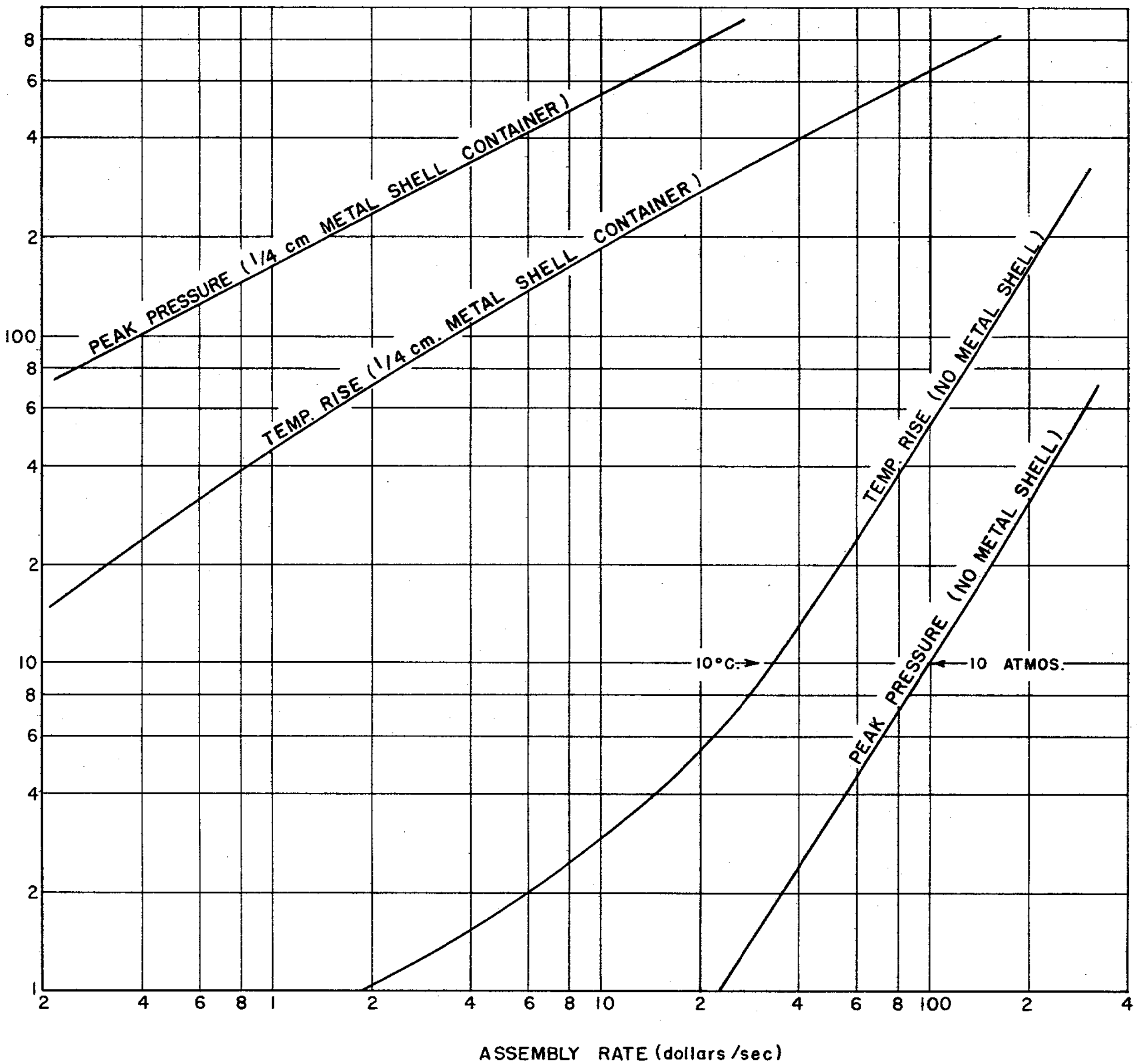

FIG. 2. Burst characteristics - Liquid assembly.

IV. EFFECT OF PARTIALLY FREE SURFACE IN SOLUTION ASSEMBLIES

As a simple geometrical model to test the effect of a partially free surface, we will choose a cylindrical liquid assembly whose faces are open and whose walls are rigidly contained (it having already been seen that such containment is approximated by a metal vessel). At the height to diameter ratio limits of zero and infinity one duplicates the cases of no shell and rigid container given in the preceding section.

The displacement vector now has the form:

$$\vec{\xi} = \vec{i}\ \xi + \vec{k}\eta$$

where $\vec{i}$ and $\vec{k}$ are unit vectors in the radial (r) and axial (z) directions respectively. With this coordinate system the expressions for the basic equations are:

Continuity equation: $-\delta\rho/\rho = \frac{\partial\xi}{\partial r} + \frac{1}{r}\xi + \frac{\partial\eta}{\partial z}$ IV-1

Equation of state: $P = \frac{\partial P}{\partial T} T - K\left\{\frac{\partial\xi}{\partial r} + \frac{1}{r}\xi + \frac{\partial\eta}{\partial z}\right\}$ IV-2

Equation of motion: $\rho\frac{\partial^2\xi}{\partial t^2} = -\frac{\partial P}{\partial r}$; $\rho\frac{\partial^2\eta}{\partial t^2} = -\frac{\partial P}{\partial z}$ IV-3

Energy conservation: $\emptyset = C_v T$ IV-4

Associated with these equations is the auxiliary condition

$\frac{\partial\xi}{\partial z} = -\frac{\partial\eta}{\partial r}$ which implies no rotation of the liquid.

With the energy release $\emptyset$ having the prescribed form

$e^{\alpha t} J_o(k_1 r) \cos k_2 z$, where J_o is the zeroth order Bessel function (as tabulated for example in Jahnke-Emde and to be distinguished from the previously used symbol j_o which referred to the spherical Bessel function), and the constants k_1 and k_2 are determined by the assembly shape, the forced solutions for ξ and η are:

$$k_1 \xi = \frac{k_1^2}{k_1^2 + k_2^2} \quad \frac{\phi}{1 + \alpha^2} \; J_1(k_1 r) \cos k_2 z$$

$$k_2 \eta = \frac{k_2^2}{k_1^2 + k_2^2} \quad \frac{\phi}{1 + \alpha^2} \; J_o(k_1 r) \sin k_2 z$$

where as formerly, ϕ is expressed in units of V_o^2/γ and α in units of $\omega = \sqrt{K \,(k_1^2 + k_2^2)/\rho}$.

With the boundary conditions $\xi(r_o, z) = 0$ and $P(r, z_o) = 0$, where r_o and $2z_o$ are the radius and height of the cylindrical assembly, one obtains:

$$k_1 \xi = \frac{k_1^2}{k_1^2 + k_2^2} \; \frac{\phi}{1 + \alpha^2} \left\{ J_1(k_1 r) - \frac{J_1(k_1 r_o) J_1(i\mu_o r)}{J_1(i\mu_o r_o)} \right\} \times \cos k_2 z \qquad \text{IV-5}$$

$$k_2 \eta = \frac{k_2^2}{k_1^2 + k_2^2} \; \frac{\phi}{1 + \alpha^2} \left\{ J_o(k_1 r) + \frac{k_1 J_1(k_1 r_o) J_o(i\mu_o r)}{[- i\mu_o J_1(i\mu_o r_o)]} \right\} \times \sin k_2 z \qquad \text{IV-6}$$

$$P = \frac{\alpha^2}{1 + \alpha^2} \phi \left\{ J_o(k_1 r) + \frac{k_1 J_1(k_1 r_o) J_o(i\mu_o r)}{[- i\mu_o J_1(i\mu_o r_o)]} \right\} \times \cos k_2 z \qquad \text{IV-7}$$

$$- \frac{\delta\rho}{\rho} = \frac{\phi}{1 + \alpha^2} \left\{ J_o(k_1 r) - \alpha^2 \frac{k_1 J_1(k_1 r_o) J_o(i\mu_o r)}{[- i\mu_o J_1(i\mu_o r_o)]} \right\} \times \cos k_2 z \qquad \text{IV-8}$$

In these equations $\mu_o^2 = k_2^2 + (k_1^2 + k_2^2)\,\alpha^2$ and the pressure is expressed in units of K.

If one neglects the difference between the actual and so-called nuclear dimensions of the assembly, then $J_o(k_1 r_o) = 0$ and $\sin k_2 z_o = 0$ so that $k_1 = 2.405/r_o$ and $k_2 = 1.57/z_o$. For $z_o > r_o$ and $\alpha z_o/r_o \ll 1$, which as will be seen is the interesting range, the equations for the pressure and density change may be written as:

$$p \cong \frac{\alpha^2}{1 + \alpha^2} \phi \left\{ J_o(k_1 r) + (z_o/r_o)^2 \right\} \qquad \text{IV-9}$$

$$- \frac{\delta\rho}{\rho} \cong \frac{\phi}{1 + \alpha^2} \left\{ J_o(k_1 r) - \alpha^2 (z_o/r_o)^2 \right\} \qquad \text{IV-10}$$

In these equations, the z_o/r_o terms are associated with the rigid wall condition. For $z_o < r_o$, IV-9 and IV-10 exaggerate the importance of these terms, and hence with the help of Table IV one concludes that the degree of containment represented by $z_o < r_o$ is of no importance for the listed range of assembly rates (i.e., less than 250 dollars/sec.).

Table V gives a listing of burst duration, pressure developed, and temperature rise as functions of assembly rate for the case $z_o/r_o = 6$ which corresponds to ~ 85% of the liquid surface being rigidly contained. Figure 3 gives a plot of temperature rise and pressure developed as functions of assembly rate for this case and the no metal shell case of the preceding section.

Concluding remarks. The tables and figures contained in this report have been based on the delayed critical prompt alpha values of 1.0×10^6 $\sec^{-1}$ for the metal assembly and 1.0×10^2 $\sec^{-1}$ for the liquid assemblies, these corresponding approximately to the measured values for the untamped Oralloy sphere and the Hypo Water Boiler. The dependence of burst duration, pressure developed, and temperature rise on α_R is given quite well by the following qualitative argument:

The parameters a and b in the equation $\ddot{\phi}/\dot{\phi} = at - b\phi$ contain α_R linearly so one may write

$$a = \alpha_R a_o \; ; \quad b = \alpha_R b_o$$

The equation governing burst duration is:

$$e^{\frac{1}{2}at_o^2} / \tfrac{1}{2}at_o^2 \cong \frac{2a}{b\dot{\phi}_o} \cong \text{constant}$$

and hence $at_o^2 \cong$ a constant $\equiv C$. For both metal and incompletely contained liquid assemblies $T = \phi = 2at_o/b$ and

TABLE V. Burst characteristics for an open ended cylindrical liquid assembly having a height to diameter ratio of 6 to 1.

Assembly rate (dollars /sec.)	at_o(in units of ω)	Ø joules /gm)	Temp. °C.	Pressure (atmos-pheres)	Burst duration (sec)
.256	.00135	1.5	0.35	.023	1.35
2.56	.0045	9	2.2	.89	.45
25.6	.0156	365	87	37	.156
76.8	.028	1350	322	215	.093

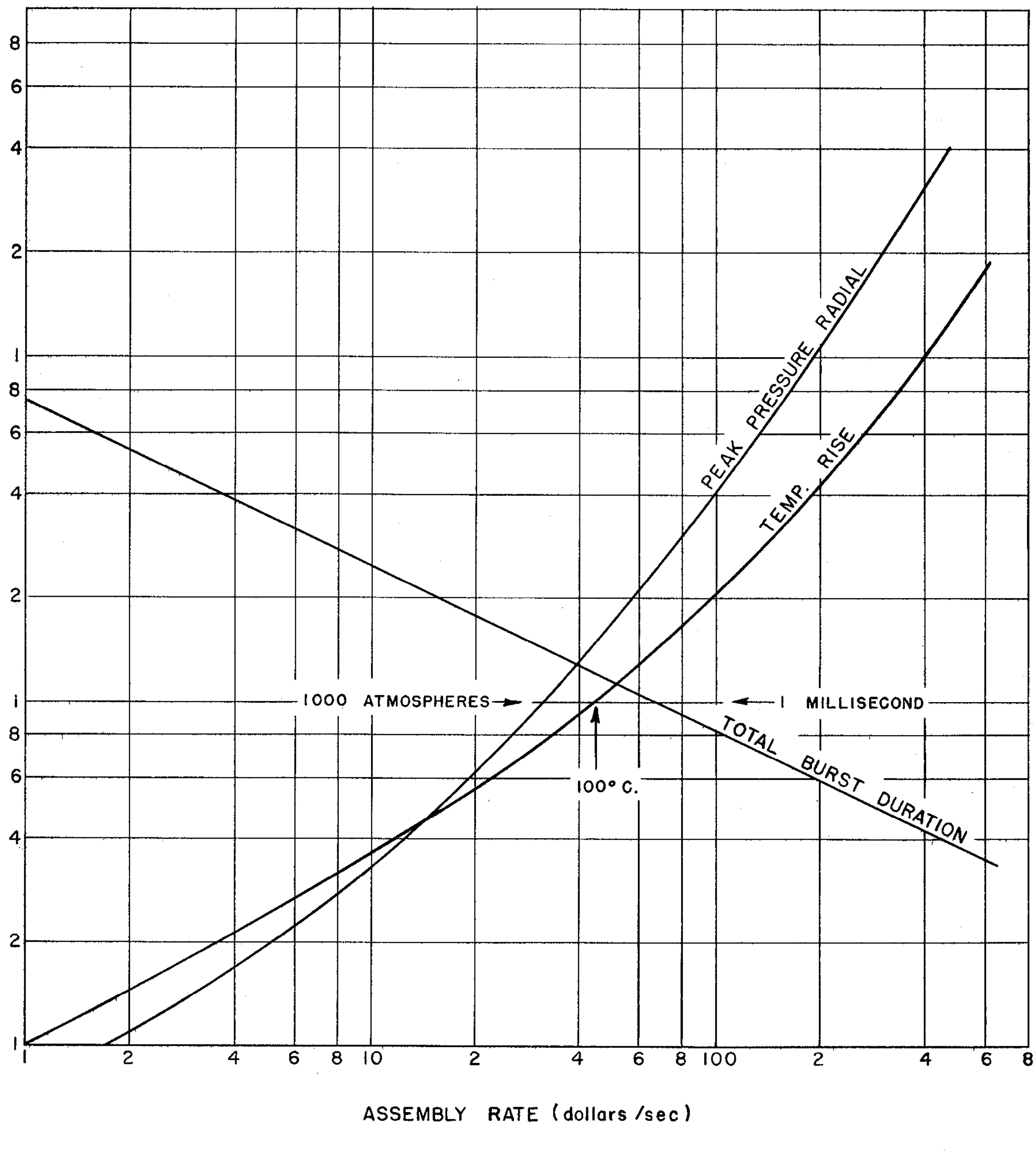

FIG. 3. Burst characteristics - Untamped metal assembly.

$P \cong \alpha^2 \emptyset = (at_o)^2 \emptyset$ where T, $\emptyset$, and P are expressed in the units $V_o^2/\gamma C_v$, V_o^2/γ , and K. Hence one obtains:

Burst duration: $t_o \cong \sqrt{C/\alpha_R a_o} \sim \alpha_R^{-1/2}$

Pressure: $P \cong 2\sqrt{C^3 a_o^3 \alpha_R / b_o^2} \sim \alpha_R^{1/2}$

Temp. rise: $V_o^2 T/\gamma C_v \cong \frac{2V_o^2}{\gamma C_v} \sqrt{\frac{Ca_o}{b_o^2 \alpha_R}} \sim \alpha_R^{-1/2}/\gamma$

With respect to the temperature rise: in a metal assembly γ is a constant and temperature $\sim \alpha_R^{-1/2}$, but in liquid assemblies the gas evolution makes γ dependent upon the temperature and pressure, specifically as:

$$\gamma \sim \frac{\delta P}{\delta T} \sim 1 + 70 \left(\frac{T_o + T}{T_o}\right) \left(\frac{P_o}{P_o + P}\right)$$

where P_o and T_o are the initial pressure and temperature (assumed equal to 1 atmosphere and 290°A.) and T and P are the temperature and pressure increases produced by the energy burst.

With respect to the initial power level: if the assembly rate is commenced from a well below critical configuration in which there is a steady source of fissions S, then as critical is approached the source fissions are chain multiplied and $\dot{\emptyset}_o \simeq S \sqrt{\frac{\pi \alpha_R}{2(\gamma f)^2 d \#/dt}}$

where γf = the reactivity change between delayed and prompt critical = effective fraction of neutrons delayed. This relation between $\dot{\phi}_o$ and S considers the multiplication of prompt neutrons alone and hence is valid only for assembly rates which cover the reactivity interval γf in a time small compared to the delayed neutron periods, - a condition which has been tacitly assumed throughout the report.