Channel Capacity

How fast can we transmit information over a communication channel?

Suppose a source sends r messages per second, and the entropy of a message is H bits per message. The information rate is $R = rH$ bits/second.

One can intuitively reason that, for a given communication system, as the information rate increases the number of errors per second will also increase. *Surprisingly, however, this is not the case*.

Shannon's theorem:

- A given communication system has a maximum rate of information C known as the **channel capacity**.
- If the information rate R is less than C, then one can approach arbitrarily small error probabilities by using intelligent coding techniques.
- To get lower error probabilities, the encoder has to work on longer blocks of signal data. This entails longer delays and higher computational requirements.

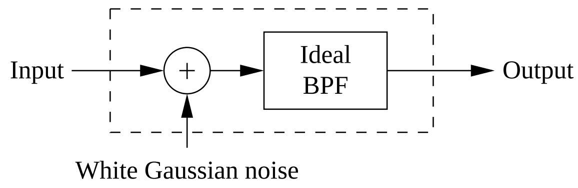
Thus, if $R \leq C$ then transmission may be accomplished without error *in the presence of noise*.

Unfortunately, Shannon's theorem is not a constructive proof — it merely states that such a coding method exists. The proof can therefore not be used to develop a coding method that reaches the channel capacity.

The negation of this theorem is also true: if $R > C$, then errors cannot be avoided regardless of the coding technique used.

1 Shannon-Hartley theorem

Consider a bandlimited Gaussian channel operating in the presence of additive Gaussian noise:



The Shannon-Hartley theorem states that the channel capacity is given by

$$C = B \log_2(1 + S/N)$$

where C is the capacity in bits per second, B is the bandwidth of the channel in Hertz, and S/N is the signal-to-noise ratio.

We cannot prove the theorem, but can partially justify it as follows: suppose the received signal is accompanied by noise with a RMS voltage of σ, and that the signal has been quantised with levels separated by $a = \lambda\sigma$. If λ is chosen sufficiently large, we may expect to be able to recognise the signal level with an acceptible probability of error. Suppose further that each message is to be represented by one voltage level. If there are to be M possible messages, then there must be M levels. The average signal power is then

$$S = \frac{M^2 - 1}{12}(\lambda\sigma)^2.$$

The number of levels for a given average signal power is therefore

$$M = \left(1 + \frac{12}{\lambda^2}\frac{S}{N}\right)^{1/2},$$

where $N = \sigma^2$ is the noise power. If each message is equally likely, then each carries an equal amount of information

$$H = \log_2 M = \frac{1}{2}\log_2\left(1 + \frac{12}{\lambda^2}\frac{S}{N}\right) \text{ bits/message.}$$

To find the information rate, we need to estimate how many messages can be carried per unit time by a signal on the channel. Since the discussion is heuristic, we note that the response of an ideal LPF of bandwidth B to a unit step has a 10–90 percent rise time of $\tau = 0.44/B$. We estimate therefore that with $T = 0.5/B \approx \tau$ we should be able to reliably estimate the level. The message rate is then

$$r = \frac{1}{T} = 2B \text{ messages/s.}$$

The rate at which information is being transferred across the channel is therefore

$$R = rH = B\log_2\left(1 + \frac{12}{\lambda^2}\frac{S}{N}\right).$$

This is equivalent to the Shannon-Hartley theorem with $\lambda = 3.5$. Note that this discussion has estimated the rate at which information can be transmitted with reasonably small error — the Shannon-Hartley theorem indicates that with sufficiently advanced coding techniques transmission at channel capacity can occur with *arbitrarily* small error.

The expression of the channel capacity of the Gaussian channel makes intuitive sense:

- As the bandwidth of the channel increases, it is possible to make faster changes in the information signal, thereby increasing the information rate.
- As S/N increases, one can increase the information rate while still preventing errors due to noise.
- For no noise, $S/N \to \infty$ and an infinite information rate is possible irrespective of bandwidth.

Thus we may trade off bandwidth for SNR. For example, if $S/N = 7$ and $B = 4$kHz, then the channel capacity is $C = 12 \times 10^3$ bits/s. If the SNR increases to $S/N = 15$ and B is decreased to 3kHz, the channel capacity remains the same.

However, as $B \to \infty$, the channel capacity does not become infinite since, with an increase in bandwidth, the noise power also increases. If the noise power spectral density is $\eta/2$, then the total noise power is $N = \eta B$, so the Shannon-Hartley law becomes

$$\begin{aligned} C &= B\log_2\left(1 + \frac{S}{\eta B}\right) = \frac{S}{\eta}\left(\frac{\eta B}{S}\right)\log_2\left(1 + \frac{S}{\eta B}\right) \\ &= \frac{S}{\eta}\log_2\left(1 + \frac{S}{\eta B}\right)^{\eta B/S}. \end{aligned}$$

Noting that

$$\lim_{x\to 0}(1+x)^{1/x} = e$$

and identifying x as $x = S/\eta B$, the channel capacity as B increases without bound becomes

$$C_\infty = \lim_{B\to\infty} C = \frac{S}{\eta}\log_2 e = 1.44\frac{S}{\eta}.$$

This gives the maximum information transmission rate possible for a system of given power but no bandwidth limitations.

The power spectral density can be specified in terms of equivalent noise temperature by $\eta = kT_{\text{eq}}$.

There are literally *dozens* of coding techniques — entire textbooks are devoted to the subject, and it is an active research subject. Obviously all obey the Shannon-Hartley theorem.

Some general characteristics of the Gaussian channel can be demonstrated. Suppose we are sending binary digits at a transmission rate equal to the channel capacity: $R = C$. If the average signal power is S, then the average energy per bit is $E_b = S/C$, since the bit duration is $1/C$ seconds.

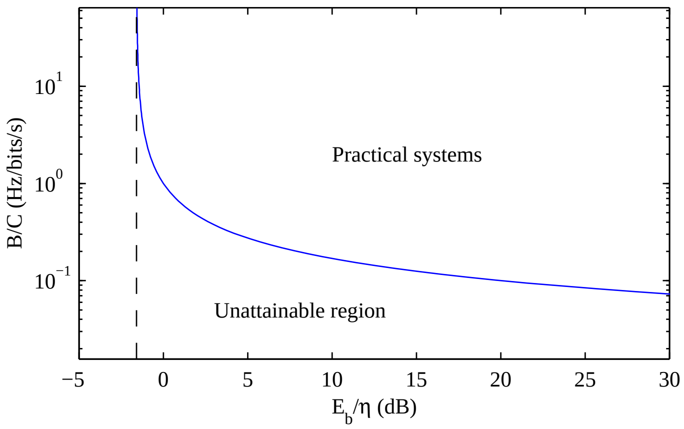
With $N = \eta B$, we can therefore write

$$\frac{C}{B} = \log_2 \left(1 + \frac{E_b}{\eta} \frac{C}{B} \right).$$

Rearranging, we find that

$$\frac{E_b}{\eta} = \frac{B}{C}(2^{C/B} - 1).$$

This relationship is as follows:



The asymptote is at $E_b/\eta = -1.59$dB, so below this value there is no error-free communication at any information rate. This is called the **Shannon limit**.