The Black Body Radiation

= Chapter 4 of Kittel and Kroemer

The Planck distribution

Derivation

Black Body Radiation

Cosmic Microwave Background

The genius of Max Planck

Other derivations

Stefan Boltzmann law

Flux => Stefan- Boltzmann

Example of application: star diameter

Detailed Balance: Kirchhoff laws

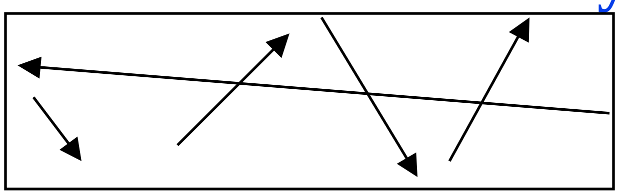
Another example: Phonons in a solid

Examples of applications

- Study of Cosmic Microwave Background
- Search for Dark Matter

The Planck Distribution

Photons in a cavity



Mode characterized by: Frequency ν; Angular frequency $\omega = 2\pi\nu$

Number of photons s in a mode => energy $\varepsilon = sh\nu = s\hbar\omega$

s is an integer ! Quantification

Similar to harmonic oscillator

Photons on same "orbital" cannot be distinguished. This is a *quantum state,* not s systems in interactions!

Occupation number

! System =1 mode , in contact with reservoir of temperature

Partition function

$$Z = \sum_{s=0}^{\infty} e^{-\frac{s\hbar\omega}{\tau}} = \frac{1}{1 - \exp\left(-\frac{\hbar\omega}{\tau}\right)}$$

Mean number of photons

$$\langle s \rangle = \sum_s sp(s) = \frac{1}{Z}\sum_s se^{-\frac{s\hbar\omega}{\tau}} = -\frac{\partial \log Z}{\partial\left(\frac{\hbar\omega}{\tau}\right)} = \frac{exp\left(-\frac{\hbar\omega}{\tau}\right)}{1 - exp\left(-\frac{\hbar\omega}{\tau}\right)}$$

***Planck Distribution

$$\langle s \rangle = \frac{1}{exp\left(\frac{\hbar\omega}{\tau}\right) - 1}$$

$$\langle \varepsilon_\omega \rangle = \frac{\hbar\omega}{exp\left(\frac{\hbar\omega}{\tau}\right) - 1}$$

Black Body Radiation

Maxwell equations in vacuum

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_o} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_o\left(\vec{j} + \varepsilon_o\frac{\partial\vec{E}}{\partial t}\right) = \mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \quad (\vec{j} = 0!)$$

Using identity $\vec{\nabla}\times\vec{\nabla}\times\vec{E} = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) - \nabla^2\vec{E}$

$$\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{(wave equation)}$$

Solutions $\vec{E} = \vec{E}_o \ \exp\left(i\left(\vec{k}.\vec{x} - \omega t\right)\right)$ with $k = \frac{\omega}{c}$ and $\vec{k}.\vec{E}_o = 0$ $\qquad p = \hbar k = \frac{\hbar\omega}{c} = \frac{\varepsilon}{c}$

=> Photon has zero mass and 2 polarizations

Radiation energy between ω and ω+dω?

State (=mode)density in phase space $\; 2\frac{d^3x d^3p}{h^3}$ integrating over angles $\frac{d^3x \ p^2dp}{\pi^2\hbar^3} = \frac{d^3x \ \omega^2 d\omega}{\pi^2 c^3}$

=> density of energy

$$u_\omega d\omega = \frac{\#\ \text{states}}{\text{unit volume}} \times \text{energy of state} \times \text{average occupation number}$$

$$u_\omega d\omega = \frac{\omega^2 d\omega}{\pi^2 c^3}\times\hbar\omega\times\frac{1}{\exp\left(\frac{\hbar\omega}{\tau}\right)-1}$$

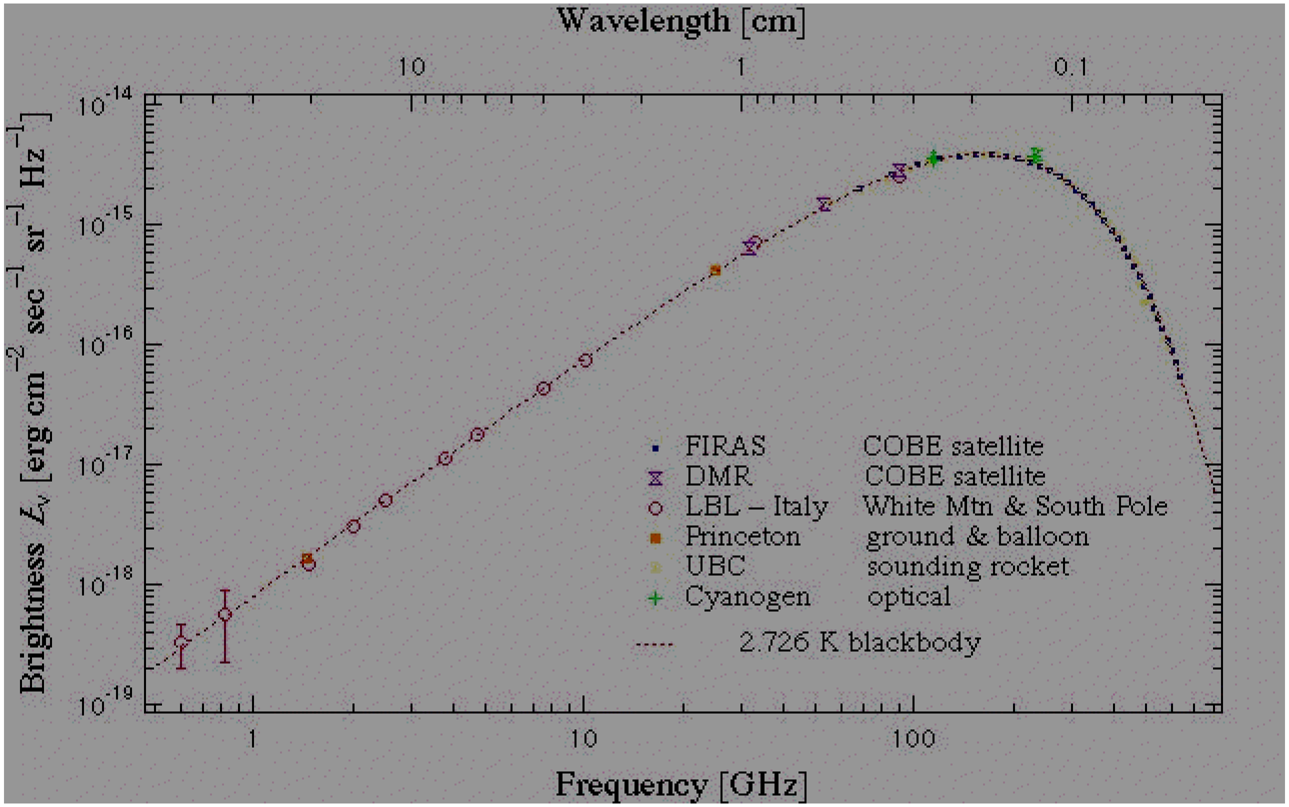
or

$$\boxed{u_\omega d\omega = \frac{\hbar\omega^3 d\omega}{\pi^2 c^3\left(\exp\left(\frac{\hbar\omega}{\tau}\right)-1\right)} = u_\nu d\nu = \frac{8\pi h\nu^3 d\nu}{c^3\left(\exp\left(\frac{h\nu}{\tau}\right)-1\right)}}$$

Cosmic Microwave Radiation

Big Bang=> very high temperatures!

When T≈3000K, p+e recombine =>H and universe becomes transparent



Conclusions:

- Very efficient thermalization
  - In particular no late release of energy
- No significant ionization of universe since!
  - e.g., photon-electron interactions= Sunyaev-Zel'dovich effect
- Fluctuations of T =>density fluctuations
  - COBE DMR result

The genius of Max Planck

Before

Physicists only considered continuous set of states (no 2nd quantization!)

Partition function of mode ω

$$Z_\omega = \int_0^\infty dx \exp\left(-\frac{\hbar\omega x}{\tau}\right) = \frac{\tau}{\hbar\omega}$$

Mean energy in mode ω

$$\langle \varepsilon_\omega \rangle = \frac{\tau^2 \partial \log Z}{\partial \tau} = \tau \qquad \text{independent of } \omega$$

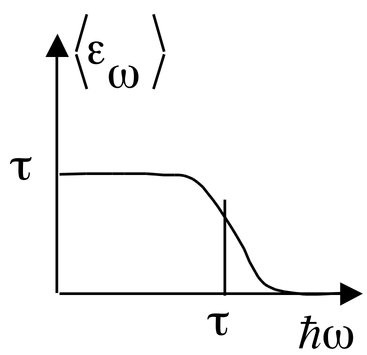
Sum on modes => infinity =“ultraviolet catastrophe”

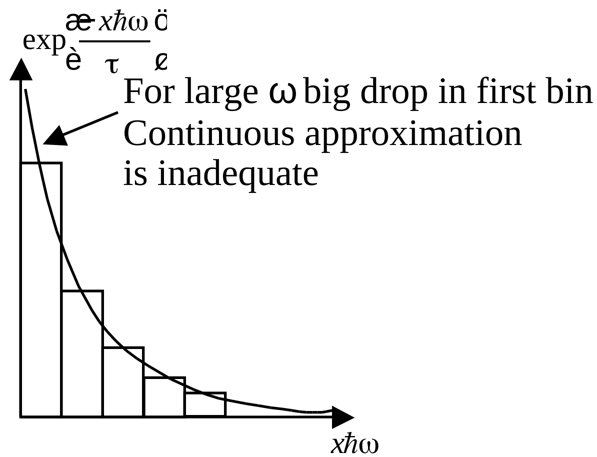
Max Planck

Quantized!

$$Z = \frac{1}{1 - \exp\left(-\frac{\hbar\omega}{\tau}\right)}$$

$$\langle \varepsilon_\omega \rangle = \frac{\hbar\omega}{\exp\left(\frac{\hbar\omega}{\tau}\right) - 1}$$





Cut off at $\hbar\omega \approx \tau$ $\qquad$ x cannot be smaller than 1!

=> finite integral

Other derivations (1)

Grand canonical method

Consider N photons $$Z = \sum_N \exp\left(\frac{(\mu - \hbar\omega)N}{\tau}\right) = \frac{1}{1 - \exp\left(\frac{\mu - \hbar\omega}{\tau}\right)}$$

=> $$\langle s \rangle = \langle N \rangle = \frac{\partial(\tau \log Z)}{\partial \mu} = \frac{1}{\exp\left(\frac{\hbar\omega - \mu}{\tau}\right) - 1}$$

What is μ? There is no exchange of photons with reservoir

(only exchange of energy) => entropy of reservoir does not change with the number of photons in the black body (at constant energy)

$$\left.\frac{\partial \sigma_R}{\partial N_{\gamma BB}}\right|_U = \frac{-\mu_\gamma}{\tau} = 0$$

Zero chemical potential! => $$\langle s \rangle = \frac{1}{\exp\left(\frac{\hbar\omega}{\tau}\right) - 1}$$

Microscopic picture

Photons are emitted and absorbed by electrons on walls of cavity

We have the equilibrium $\gamma + e \leftrightarrow e$

Special case of equilibrium (cf Kittel & Kroemer Chap. 9 p. 247)

$$A + B \leftrightarrow C$$

Equilibrium corresponds to maximum entropy $d\sigma = 0$

=>

In particular $$d\sigma|_{U,V} = \frac{\partial\sigma}{\partial N_A} dN_A + \frac{\partial\sigma}{\partial N_B} dN_B + \frac{\partial\sigma}{\partial N_C} dN_C = 0$$

$$dN_A = dN_B = -dN_C \qquad \mu_A + \mu_B - \mu_C = 0$$

$$\mu_\gamma + \mu_e - \mu_e = 0 \qquad \mu_\gamma = 0$$

this is due to the fact that the number of photons is not constant

Other derivations (2)

Microcanonical method (Done in Homework)

To compute mean number photons in one mode, consider ensemble of N oscillators at same temperature and compute total energy U

=> $$\langle \varepsilon \rangle = \lim_{N\to\infty} \frac{U}{N} \qquad \langle s \rangle = \frac{\langle \varepsilon \rangle}{\hbar\omega} = \lim_{N\to\infty} \frac{U}{N\hbar\omega}$$

Microcanonical= compute entropy $\sigma(U)$ with $U = n\hbar\omega$ (n photons in combined system)

and define temperature at equilibrium as $$\frac{1}{\tau} = \frac{\partial\sigma}{\partial U} \qquad \tau(U) \qquad U(\tau) \qquad \langle s\rangle(\tau)$$

We have to compute the multiplicity g(n,N) of number of states with energy U= number of combinations of N positive integers such that their sum is n

$$\sum_{i=1}^{N} n_i = n = \frac{U}{\hbar\omega} \qquad n_i \geq 0$$

Same problem as coefficient of t^n in expansion (cf Kittel chap. 1) $$\left(\sum_{m=0}^{\infty} t^m\right)^N = \left(\frac{1}{1-t}\right)^N = \sum_n g(n,N)\,t^n$$

=> $$g(n,N) = \frac{1}{n!}\frac{\partial^n}{\partial t^n}\left(\frac{1}{1-t}\right)^N\Bigg|_{t=0} = \frac{1}{n!} N\ (N+1)\ (N+2)\ \ (N+n-1) = \frac{(N+n-1)!}{n!\,(N-1)!}$$

Using Stirling approximation

$$\sigma(n) = log(g(n,N)) \approx N\,log\left(1+\frac{n}{N}\right) + n\,log\left(1+\frac{N}{n}\right) = N\,log\left(1+\frac{U}{\hbar\omega N}\right) + \frac{U}{\hbar\omega}\,log\left(1+\frac{N\hbar\omega}{U}\right)$$

=> $$\frac{1}{\tau} = \frac{\partial\sigma}{\partial U} = \frac{1}{\hbar\omega}\,log\left(1+\frac{N\hbar\omega}{U}\right) = \frac{1}{\hbar\omega}\,log\left(1+\frac{1}{\langle s\rangle}\right)$$ or $$\langle s\rangle = \frac{1}{exp\left(\frac{\hbar\omega}{\tau}\right)-1}$$

Counting Number of States

From first principles (Kittel-Kroemer)

In cubic box of side L (perfectly conductive=>E perpendicular to surface)

$$E_x = E_{x_o} \sin\omega t \cos\left(n_x \frac{\pi x}{L}\right)\sin\left(n_y \frac{\pi y}{L}\right)\sin\left(n_z \frac{\pi z}{L}\right) \quad B_x = B_{x_o} \cos\omega t \sin\left(n_x \frac{\pi x}{L}\right)\cos\left(n_y \frac{\pi y}{L}\right)\cos\left(n_z \frac{\pi z}{L}\right)$$

$$E_y = E_{y_o} \sin\omega t \sin\left(n_x \frac{\pi x}{L}\right)\cos\left(n_y \frac{\pi y}{L}\right)\sin\left(n_z \frac{\pi z}{L}\right) \quad B_y = B_{y_o} \cos\omega t \cos\left(n_x \frac{\pi x}{L}\right)\sin\left(n_y \frac{\pi y}{L}\right)\cos\left(n_z \frac{\pi z}{L}\right)$$

$$E_z = E_{z_o} \sin\omega t \sin\left(n_x \frac{\pi x}{L}\right)\sin\left(n_y \frac{\pi y}{L}\right)\cos\left(n_z \frac{\pi z}{L}\right) \quad B_z = B_{z_o} \cos\omega t \cos\left(n_x \frac{\pi x}{L}\right)\cos\left(n_y \frac{\pi y}{L}\right)\sin\left(n_z \frac{\pi z}{L}\right)$$

$$\vec{B}_o = \frac{\pi}{\omega L}\vec{n} \times \vec{E}_o$$

$$B_o^2 = \frac{n^2\pi^2}{\omega^2 L^2}E_o^2$$

with the wave equation constraint $\frac{\pi^2}{L^2}\left(n_x^2 + n_y^2 + n_z^2\right) = \frac{\omega^2}{c^2} \Leftrightarrow n^2 = \frac{\omega^2 L^2}{\pi^2 c^2}$

=> ω has specific values! ("1st quantification")

2 degrees of freedom (massless spin 1 particle) $\vec{\nabla}\cdot\vec{E} = 0 \Leftrightarrow \vec{n}\cdot\vec{E} = 0 \Leftrightarrow n_x E_{x_o} + n_y E_{y_o} + n_z E_{z_o} = 0$

("2nd quantification") $U = \left\langle \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right)\right\rangle_{\text{space,time}} = \left\langle \frac{\varepsilon_o}{2}\left(E^2 + c^2B^2\right)\right\rangle = \frac{\varepsilon_o E_o^2}{16} = s\hbar\omega$ sum of 4 $\cos^2$

=> number of states in mode ω is 2xnumber of integers satisfying wave equation constraints

Sum on states can be replaced by integral on quadrant of sphere of radius squared $n_x^2 + n_y^2 + n_z^2 = \frac{\omega^2 L^2}{\pi^2 c^2}$

Energy in ω ω+dω

$$U_\omega d\omega = 2\sum_{n_x,n_y,n_z} \frac{\hbar\omega}{\exp\left(\frac{\hbar\omega}{\tau}\right) - 1} = 2\int_{\text{quadrant}} \frac{\hbar\omega}{\exp\left(\frac{\hbar\omega}{\tau}\right) - 1} d^3n$$

$$U_\omega d\omega = 2\,\frac{4\pi}{8} n^2 dn \frac{\hbar\omega}{\exp\left(\frac{\hbar\omega}{\tau}\right) - 1} = \frac{\pi\omega^2 L^3}{\pi^3 c^3}\frac{\hbar\omega\, d\omega}{\exp\left(\frac{\hbar\omega}{\tau}\right) - 1}$$

$$u_\omega d\omega = \frac{U_\omega}{L^3} d\omega = \frac{1}{\pi^2 c^3}\frac{\hbar\omega^3 d\omega}{\exp\left(\frac{\hbar\omega}{\tau}\right) - 1}$$

Fluxes

Energy density

So far energy density integrated over solid angle. If we are interested in energy density traveling traveling in a certain direction, isotropy implies

$$u_\omega(\Omega)\,d\omega\, d\Omega = \frac{\hbar\omega^3 d\omega\, d\Omega}{4\pi^3 c^3\left(\exp\left(\frac{\hbar\omega}{\tau}\right)-1\right)}$$

Note if we use ν instead of ω

$$u_\nu(\Omega)\,d\nu\, d\Omega = \frac{2h\nu^3 d\nu\, d\Omega}{c^3\left(\exp\left(\frac{h\nu}{\tau}\right)-1\right)}$$

Flux in a certain direction

(Energy /unit time, area, solid angle,frequency)

$$I_\nu\, d\nu\, dA\, d\Omega = c u_\nu\, d\nu\, dA\, d\Omega = \frac{2h\nu^3 d\nu\, dA\, d\Omega}{c^2\left(\exp\left(\frac{h\nu}{\tau}\right)-1\right)}$$

Flux through a fixed opening

(Energy /unit time, area,frequency)

$$J_\nu\, d\nu\, dA = d\nu\, dA \int_0^{2\pi} d\varphi \int_0^1 I_\nu \cos\theta \; d\cos\theta = \frac{2\pi h\nu^3 d\nu\, dA}{c^2\left(\exp\left(\frac{h\nu}{\tau}\right)-1\right)} = \frac{c}{4} u_\nu\, d\nu\, dA$$

Stefan-Boltzmann Law

Total Energy Density***

Integrate on ω

$$\int_0^\infty u_\omega \, d\omega = \int_0^\infty \frac{1}{\pi^2 c^3} \frac{\hbar\omega^3}{\exp\left(\frac{\hbar\omega}{\tau}\right) - 1} d\omega$$

Change of variable

$$\omega \to x = \frac{\hbar\omega}{\tau}$$

$$\int_0^\infty u_\omega \, d\omega = \frac{\tau^4}{\hbar^3 \pi^2 c^3} \int_0^\infty \frac{x^3 dx}{e^x - 1}$$

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$$

$$u = \frac{\pi^2}{15\hbar^3 c^3} \tau^4 = a_B T^4$$

with $a_B = \dfrac{\pi^2 k_B^4}{15\hbar^3 c^3}$

Total flux through an aperture***

Integrate J_ν

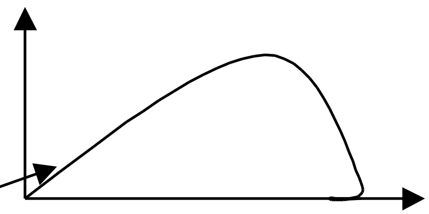
= multiply above result by c/4

$$J = \frac{\pi^2}{60\hbar^3 c^2} \tau^4 = \sigma_B T^4$$

with $\sigma_B = \dfrac{\pi^2 k_B^4}{60\hbar^3 c^2} = 5.67 \; 10^{-8} \;\; \mathrm{W/m^2/K^4}$

Stefan-Boltzmann constant!

Spectrum at low frequency



Rayleigh-Jeans region (= low frequency)

if $\hbar\omega << \tau$ $\quad \langle \varepsilon_\omega \rangle \approx \tau$

$$u_\omega d\omega \approx \tau \frac{\omega^2 d\omega}{\pi^2 c^3} \quad u_\nu d\nu \approx \tau \frac{8\pi\nu^2 d\nu}{c^3}$$

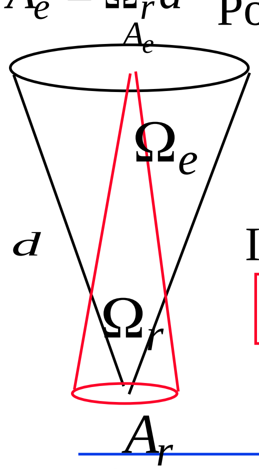
Power / unit area/solid angle/unit frequency
=Brightness

$$I_\nu d\nu dA d\Omega \approx \tau \frac{2\nu^2 d\nu dA d\Omega}{c^2} = \tau \frac{2 d\nu dA d\Omega}{\lambda^2}$$

Power emitted / unit (fixed) area

$$J_\omega d\omega = J_\nu d\nu \approx \tau \frac{\omega^2 d\omega}{4\pi^2 c^2} = \tau \frac{2\pi d\nu}{\lambda^2}$$

$A_e = \Omega_r d^2$ Power received from a *diffuse* source



$$\Omega_e = \frac{A_r}{d^2}$$

$$\frac{dP}{d\nu} = I_\nu \Omega_e A_e = \frac{2\tau\Omega_e A_e}{\lambda^2} = \frac{2\tau\Omega_r A_r}{\lambda^2} = 2\tau$$

If diffraction limited:

$$\boxed{\Omega_r A_r = \lambda^2}$$

Detected Power (1 polarization) $\boxed{\frac{dP}{d\nu}(1\text{polarization}) = \tau}$

Antenna temperature $T_A = \frac{1}{k_B}\frac{dP}{d\nu}(1\text{polarization})$

Applications

Many!

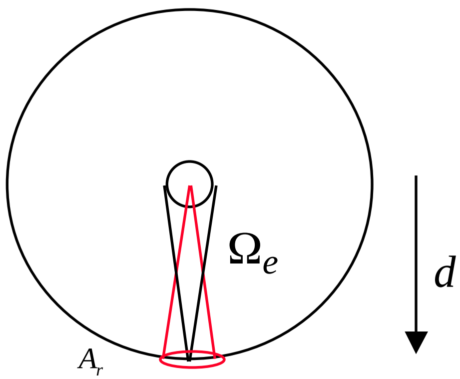
e.g. Star angular diameter

Approximately black body and spherical!

Spectroscopy =>Effective temperature

Apparent luminosity l =power received per unit area

$$l = \frac{L\Omega_e}{4\pi}\frac{1}{A_r} = \frac{L}{4\pi d^2}$$



$$\Omega_e = \frac{A_r}{d^2}$$

But power output $L = 4\pi r^2 \sigma_B T_{\text{eff}}^4$

$$l = \frac{r^2}{d^2}\sigma_B T_{\text{eff}}^4$$

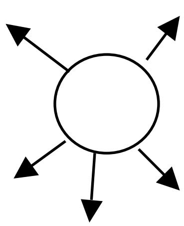
$$\text{angular diameter} = \frac{r}{d} = \sqrt{\frac{l}{\sigma_B T_{\text{eff}}^4}}$$

=> Baade-Wesserlink distance measurements of varying stars $\frac{dr}{dt} \approx \frac{dr_{//}}{dt}$ given by Doppler shift

- Oscillating stars (Cepheids, RR Lyrae)
- Supernova

Luminosity/temperature variation

$$\frac{d\theta}{dt} = \frac{d}{dt}\sqrt{\frac{l}{\sigma_B T_{\text{eff}}^4}} = \frac{1}{d}\frac{dr}{dt}$$



Doppler

velocity $v_{//} = \frac{\Delta\lambda}{\lambda} c = \frac{dr_{//}}{dt}$

If spherical expansion $\frac{dr_{//}}{dt} = \frac{dr}{dt}$

$$d = \frac{\frac{dr_{//}}{dt}}{\frac{d\theta}{dt}}$$

Entropy, Number of photons

Entropy

- *Method 1* (Kittel) use thermodynamic identity+ 3rd law at constant volume

$$dU = \tau d\sigma$$

$$\left.\frac{d\sigma}{d\tau}\right|_V = \frac{\frac{dU}{d\tau}}{\tau} = \frac{\frac{du}{d\tau}V}{\tau} = \frac{4\pi^2 V}{15\hbar^3 c^3}\tau^2$$

$$\text{Third law: } \sigma = \int_0^\tau \frac{d\sigma}{d\tau} = \frac{4\pi^2 V}{45\hbar^3 c^3}\tau^3 = \frac{4}{3}\frac{a_B}{k_B}VT^3$$

- *Method 2*: sum of entropy of each mode

$$\sigma_\omega = -\sum_s p_s \log p_s = \log Z_\omega + \frac{\langle s\rangle \hbar\omega}{\tau} = -\log\left(1 - \exp\left(\frac{-\hbar\omega}{\tau}\right)\right) + \frac{\hbar\omega}{\tau\left(\exp\left(\frac{\hbar\omega}{\tau}\right) - 1\right)}$$

Remembering density of states

$$\sigma = \int \sigma_\omega d^3x \frac{\omega^2 d\omega}{\pi^2 c^3} = V\left(\frac{1}{3}\frac{u}{\tau} + \frac{u}{\tau}\right) \text{ (integrating first term by part)}$$

Finally

$$\sigma = \frac{4}{3}\frac{u}{\tau}V = \frac{4}{3}\frac{a_B}{k_B}VT^3$$

Number of photons

Starting from

$$\langle s\rangle = \frac{1}{\exp\left(\frac{\hbar\omega}{\tau}\right) - 1} \qquad N_\gamma = \frac{V}{\pi^2 c^3}\int_0^\infty \frac{\omega^2 d\omega}{\left(\exp\left(\frac{\hbar\omega}{\tau}\right) - 1\right)}$$

Classical integral

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1}dx = (s-1)!\zeta(s) \text{ with } \zeta(3) \approx 1.22$$

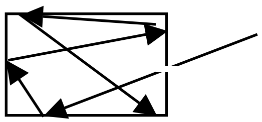
=>

$$N_\gamma = \frac{V 2\zeta(3)}{\pi^2 c^3 \hbar^3}\tau^3 = \frac{30\zeta(3)}{\pi^4}\frac{a_B}{k_B}VT^3 \approx 0.37\frac{a_B}{k_B}VT^3$$

Note that entropy is proportional to number of photons! $\sigma \approx 3.6 N_\gamma$

Detailed Balance: Kirchhoff laws

Definition: A body is black if it absorbs all electromagnetic radiation incident on it



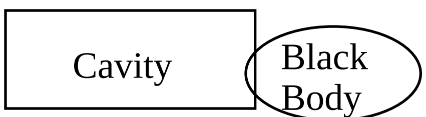
Usually true only in a range of frequency

e.g. A cavity with a small hole appears black to the outside

Detailed balance: in thermal equilibrium, power emitted= power received!

Otherwise temperature would change!

Consequence: The spectrum of radiation emitted by a black body is the "black body" spectrum calculated p. 4.



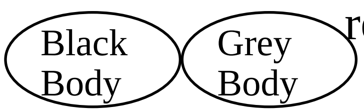
Proof : $\frac{c}{4} A u_{BB} = \frac{c}{4} A u_{cavity}$ $\quad u_{BB} = u_{cavity}$

We could have inserted a filter $\quad u_{BB}(\omega)d\omega = u_{cavity}(\omega)d\omega$

Absorptivity, Emissivity:

Absorptivity =fraction of radiation absorbed by body

Emissivity = ratio of emitted spectral density to black body spectral density.



received power = $a(\omega)u_{BB}(\omega)d\omega$ = emitted power = $e(\omega)u_{BB}(\omega)d\omega$

Kirchhoff: Emissivity=Absorptivity $a(\omega) = e(\omega)$

Applications: Johnson noise of a resistor at temperature T

$$\frac{d\langle V_{\text{noise}}^2\rangle}{d\nu} = 4k_B TR$$

We will come back to this !

Rayleigh − Jeans => Spectral density = $k_B T$

Superinsulation (shinny)

Phonons in a solid

Phonons:

Quantized vibration of a crystal

described in same way as photons $\quad \varepsilon = \hbar\omega \quad p = \hbar k = \dfrac{\hbar\omega}{c_s}$

If s phonons in a mode $\quad \varepsilon_s = s\hbar\omega$

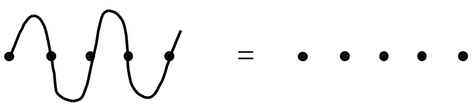
Velocity

mean occupancy $\quad \langle s \rangle = \dfrac{1}{\exp\left(\dfrac{\hbar\omega}{\tau}\right) - 1}$

Sum on modes (3 polarizations)

$$3\frac{d^3x\, d^3p}{h^3} \quad \text{integrating over angles} \quad 3\frac{d^3x \quad p^2 dp}{\pi^2 \hbar^3} = d^3x \underbrace{\frac{3\omega^2 d\omega}{\pi^2 c_s^3}}_{D(\omega)\, d\omega}$$

But wave length smaller than lattice spacing is meaningless

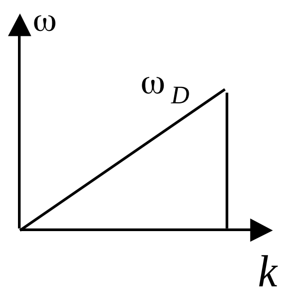


$|k_x| < \dfrac{\pi}{a}$ where a is the lattice spacing $\quad$ maximum energy ω_D

<=> solid is finite: If N atoms each with 3 degrees of freedom, at most 3N modes

$N_{\text{atoms}} = \left(\dfrac{L}{a}\right)^3$ (cubic lattice) $\quad \int_0^{\omega_D} V D(\omega)\, d\omega = 3N$ where $D(\omega)$ is the density of states

Debye approximation: • isotropic



• $k = \dfrac{\omega}{c_s}$ or $p = \dfrac{\varepsilon}{c_s}$ with velocity c_s = constant

Phonons in a Solid

Debye law

$$\int_0^{\omega_D} VD(\omega)d\omega = \int_0^{\omega_D} V\frac{3\omega^2}{2\pi^2 c_s^3}d\omega = 3N \qquad \omega_D = \left(\frac{6N}{V}\pi^2 c_s^3\right)^{\frac{1}{3}} = \left(6\pi^2\right)^{\frac{1}{3}}\frac{c_s}{a}$$

$$U = \int d^3x\int_0^{k_D}\frac{\hbar\omega}{\left(\exp\left(\frac{\hbar\omega}{\tau}\right)-1\right)}\frac{3k^2dk}{2\pi^2} = V\int_0^{\omega_D}\frac{\hbar\omega}{\left(\exp\left(\frac{\hbar\omega}{\tau}\right)-1\right)}\frac{3\omega^2 d\omega}{2\pi^2 c_s^3}$$

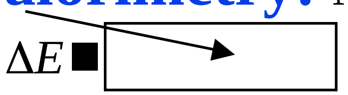
for $\tau \ll \omega_D$ $\quad U = \frac{3\tau^4}{2\hbar^3\pi^2 c_s^3}V\int_0^\infty\frac{x^3dx}{e^x-1} = \frac{\pi^2}{10\hbar^3 c_s^3}V k_B^4 T^4$

Introducing the Debye temperature $T_D = \theta = \frac{\hbar c_s}{k_B}\left(6\pi^2\frac{N}{V}\right)^{1/3} = \frac{\hbar\omega_D}{k_B}$

$$U = \frac{3}{5}\pi^4 k_B N\frac{T^4}{T_D^3} \qquad C_V = \frac{12}{5}\pi^4 k_B N\left(\frac{T}{T_D}\right)^3 \qquad \sigma = \frac{12}{15}\pi^4 k_B N\left(\frac{T}{T_D}\right)^3$$

Applications

Calorimetry: Measure energy deposition by temperature rise

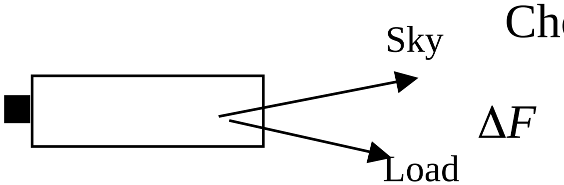


$$\Delta T = \frac{\Delta E}{C}$$ need small C

Heat Capacity

Bolometry: Measure energy flux by temperature rise

Chopping e.g., between sky and calibration load



$$\Delta T = \frac{\Delta F}{G}$$ need small G but time constant $= \frac{C}{G}$ limited by stability small heat capacity C

Heat Conductivity

Very sensitive!

Heat capacity goes to zero at low temperature $C \approx T^3$

Fluctuations

$$\sigma_{\Delta E} = \sqrt{k_B T^2 C} \quad T^{\frac{5}{2}} M^{\frac{1}{2}}$$

Wide bandwidth (sense every frequency which couples to bolometer)

Study of cosmic microwave background

Bolometers 4K → 300mK → 100mK

Search for dark matter particles 170g at 20mK