A Gallery of de Rham Curves

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Abstract

The de Rham curves are a set of fairly generic fractal curves exhibiting dyadic symmetry. Described by Georges de Rham in 1957[3], this set includes a number of the famous classical fractals, including the Koch snowflake, the Peano space-filling curve, the Cesàro-Faber curves, the Takagi-Landsberg[4] or blancmange curve, and the Lévy C-curve. This paper gives a brief review of the construction of these curves, demonstrates that the complete collection of linear affine deRham curves is a five-dimensional space, and then presents a collection of four dozen images exploring this space.

These curves are interesting because they exhibit dyadic symmetry, with the dyadic symmetry monoid being an interesting subset of the group $GL(2,\mathbb{Z})$. This is a companion article to several others[5][6] exploring the nature of this monoid in greater detail.

1 Introduction

In a classic 1957 paper[3], Georges de Rham constructs a class of curves, and proves that these curves are everywhere continuous but are nowhere differentiable (more precisely, are not differentiable at the rationals). In addition, he shows how the curves may be parameterized by a real number in the unit interval. The construction is simple. This section illustrates some of these curves.

Consider a pair of contracting maps of the plane $d_0 : \mathbb{R}^2 \to \mathbb{R}^2$ and $d_1 : \mathbb{R}^2 \to \mathbb{R}^2$. By the Banach fixed point theorem, such contracting maps should have fixed points p_0 and p_1. Assume that each fixed point lies in the basin of attraction of the other map, and furthermore, that the one map applied to the fixed point of the other yields the same point, that is,

$$d_1(p_0) = d_0(p_1) \tag{1}$$

These maps can then be used to construct a certain continuous curve between $p_0$and p_1. This is done by repeatedly composing together the maps d_0 and d_1 according to the paths of an infinite binary tree, or, equivalently, according to the elements of a Cantor set. This may be done in a very concrete fashion, by referring to the expansion in binary digits of a real number x:

$$x = \sum_{k=1}^{\infty} \frac{b_k}{2^k}$$

where each of the binary digits b_k is 0 or 1. The de Rham curve is then a map characterized by the continuous parameter x:

$$d_x = d_{b_1} \circ d_{b_2} \circ \ldots \circ d_{b_k} \circ \ldots$$

The above map will take points in the common basin of attraction of the two maps, down to a single point. De Rham provides a simple proof that the resulting set of points form a continuous curve as a function of x, and that furthermore, this function is not differentiable in any conventional sense.

1.1 Examples

De Rham provides several examples. Let $z = u + iv$ and $a \in \mathbb{C}$ be a constant such that $|a| < 1$ and $|a-1| < 1$. Then consider the maps

$$d_0(z) = az$$

and

$$d_1(z) = a + (1-a)z$$

These two maps clearly have fixed points at $z = 0$ and $z = 1$, respectively. The generated curve is is the non-differentiable curve of Cesàro and Faber, now known more generally as the Lévy C-curve, especially when $a = 0.5 + i0.5$. See figures 1 and **??**.

Written as affine transformations, the two transforms can be expressed as

$$d_0(u,v) = \left(\begin{array}{c} 1 \\ u' \\ v' \end{array} \right) = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \alpha & -\beta \\ 0 & \beta & \alpha \end{array} \right) \left(\begin{array}{c} 1 \\ u \\ v \end{array} \right)$$

and

$$d_1(u,v) = \left(\begin{array}{c} 1 \\ u' \\ v' \end{array} \right) = \left(\begin{array}{ccc} 1 & 0 & 0 \\ \alpha & 1-\alpha & \beta \\ \beta & -\beta & 1-\alpha \end{array} \right) \left(\begin{array}{c} 1 \\ u \\ v \end{array} \right)$$

where $z = u + iv$ and $a = \alpha + i\beta$.

The Takagi curve analyzed in the companion paper[6] can be generated in the same way, using

$$d_0 = L_3 = g_3 = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 1 & w \end{array} \right) \text{ and } d_1 = R_3 = r_3 g_3 r_3 = \left(\begin{array}{ccc} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & -1 & w \end{array} \right)$$

This curve is illustrated in figure 2.

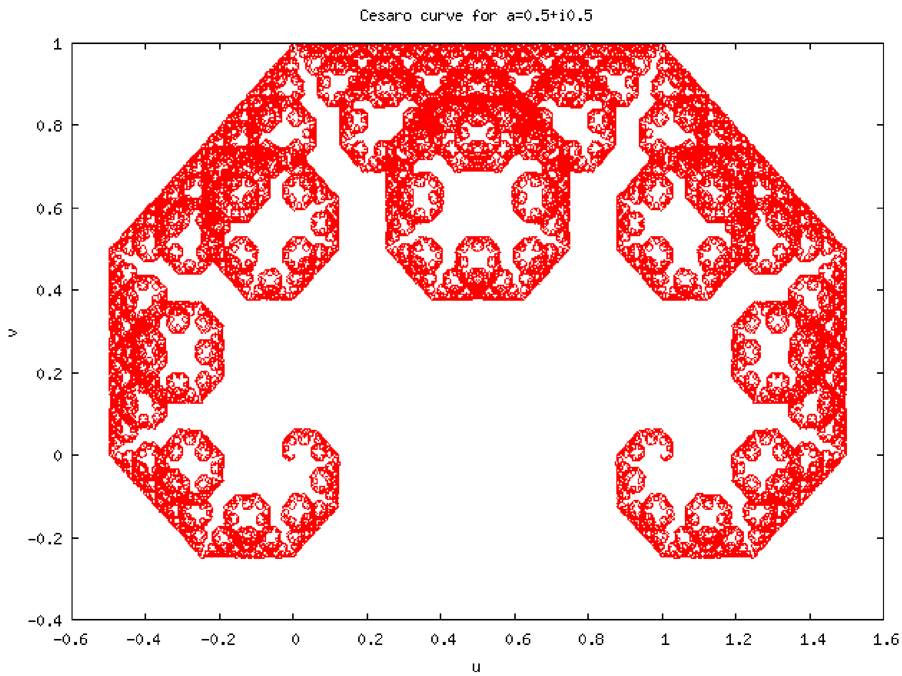
The Koch and Peano curves are similarly obtained, by introducing a mirror reflection through the complex conjugate:

$$d_0(z) = a\bar{z}$$

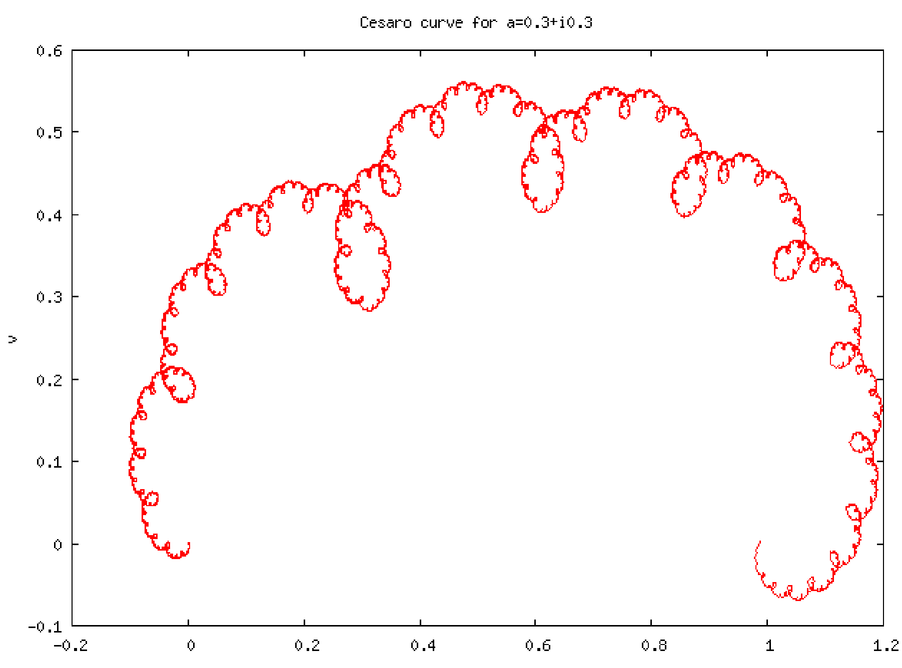
and

$$d_1(z) = a + (1-a)\bar{z}$$

Figure 1: Cesàro Curve

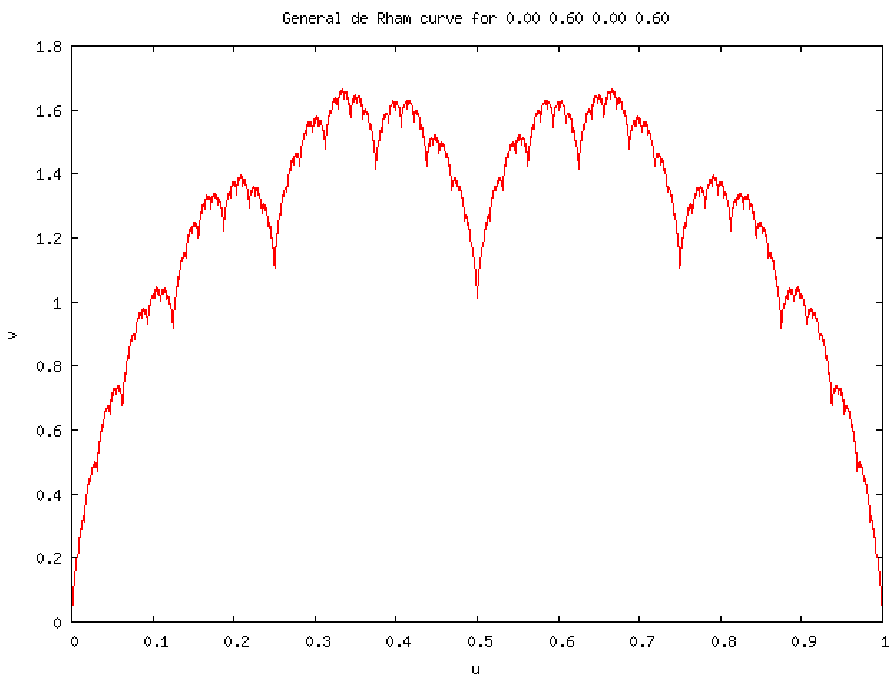


The Cesàro curve, graphed for the value of $a = (1+i)/2$. Note that this figure is more commonly known as the Lévy C-curve.



The Cesàro curve, graphed for the value of $a = 0.3 + i0.3$. The real parameter shifts the symmetry point: thus the biggest loop is located at 0.3 in this picture, instead of being located at 0.5 as in the Lévy curve. The imaginary parameter provides a "strength" of the non-differentiability, playing a role similar to the w parameter in the Blancmange curve.

Figure 2: Takagi Curve



The Takagi or blancmange curve, corresponding to a value of $w = 0.6$.

Expressed in terms of affine left and right matrices, these are:

$$d_0 = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \beta & -\alpha \end{array} \right) \text{ and } d_1 = \left(\begin{array}{ccc} 1 & 0 & 0 \\ \alpha & 1-\alpha & -\beta \\ \beta & -\beta & \alpha-1 \end{array} \right)$$

The classic Koch snowflake is regained for $a = \alpha + i\beta = 1/2 + i\sqrt{3}/6$ and the Peano curve for $a = (1+i)/2$. Values intermediate between these two generate intermediate curves, as shown in figure 3 and **??**.

Yet another example of a curve generated by means of the de Rham construction is the Minkowski question mark function[5], which is given by the Möbius functions

$$d_0(z) = z/(z+1)$$

and

$$d_1(z) = 1/(z+1)$$

Expressed as the usual 2x2 matrix representations for Möbius transforms, these correspond to the generators L and R of the Stern-Brocot tree[5, 2]. More precisely, the generated function is actually half the inverse: $d_x = ?^{-1}(2x)$.

From the construction properties, it should now be clear that this generalized de Rham curve construction has the same set of modular-group self-similarities; this essentially follows from the self-similarity properties of the Cantor polynomials. That is, given a contracting group element $\gamma = g^{a_1} r g^{a_2} r g^{a_3} r ... r g^{a_N} \in GL(2,\mathbb{Z})$, one defines its action in the canonical way, on the parameter space, as an action on dyadic intervals: thus

$$gd_x = d_{x/2}$$

and

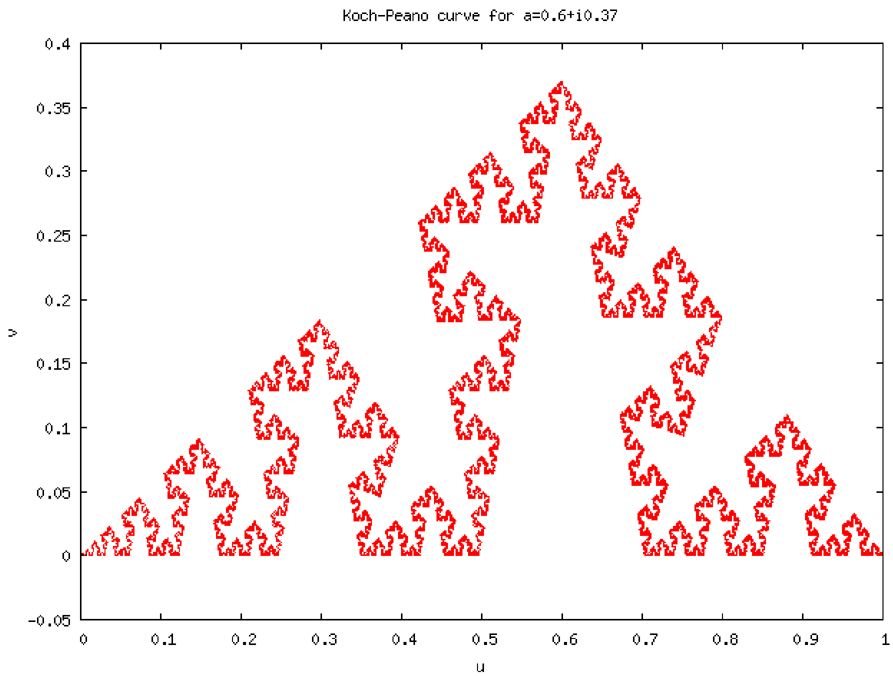
$$sd_0 = d_1 \qquad d_0 = s^{-1} d_1$$

Note that the above is not just a statement about some particular value of x, but is rather a statement that holds true for the entire range of parameters $x \in [0,1]$; it is a statement of the self-similarity properties of the curve. In the case of the Koch and Lévy curves, both g and s, and thus any contracting elements γ are expressible as linear affine transformations on the two-dimensional plane. This is essentially an expression of a known result from the theory of iterated function systems (IFS)[1]: these figures are obtainable by iterating on a pair of specific affine transforms.

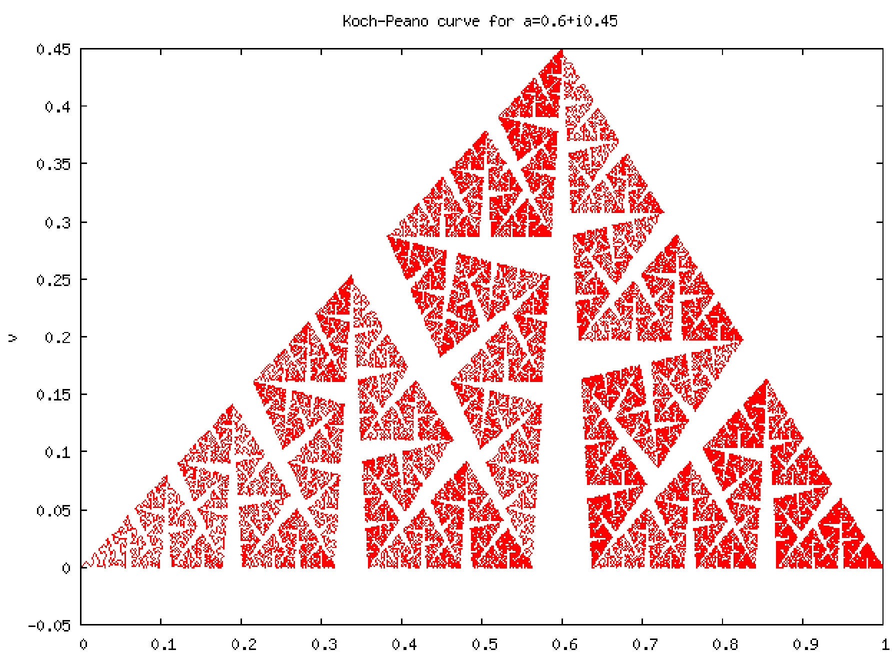
Homework: Write down an explicit expression for a general γ for the Koch and Lévy curves.

The lesson to be learned here bears stating clearly: ***every point on the above-mentioned curves can be uniquely labelled by a real number. The labelling is not abstract, but concrete. The fractal self-similarity of the curves are in unique correspondence to the contracting monoid of*** $GL(2,\mathbb{Z})$***. To every element of the contracting monoid, a unique non-degenerate mapping of the plane can be given that exactly maps the curve into a self-similar subset of itself. The mapping is continuous, and can be expressed in concrete form.***

Figure 3: Koch Curve



The Koch snowflake curve, constructed for $a = 0.6 + i0.37$. The classic, hexagonal-symmetry curve is regained by setting $a = 0.5 + i\sqrt{3}/6$, which centers the big point at 1/2, and opens the base of the point to run between 1/3 and 2/3'rds.



The Koch curve, for $a = 0.6 + i0.45$. The classic Peano space-filling curve is regained for $a = (1+i)/2$.

1.2 The Total Number of Linear, Planar Dyadic Fractal Curves

The above exposition, in terms of the action of left and right affine transformations, indicates that it is possible to count the total number of uniquely distinct dyadic planar fractal curves, and to classify them into families. The general linear, dyadic planar fractal curve is given by iterating on

$$d_0 = \begin{pmatrix} 1 & 0 & 0 \\ a & b & c \\ d & e & f \end{pmatrix} \text{ and } d_1 = \begin{pmatrix} 1 & 0 & 0 \\ h & j & k \\ l & m & n \end{pmatrix}$$

where $a, b, \ldots, n$ are taken as real numbers. From this general set, one wants to exclude the cases which are rotated, translated, scaled or squashed versions of one another. The general set appears to have twelve free parameters, from which should be excluded 1 (for rotations) + 2 (for translations) + 2 (for scaling) + 1(for shearing) = 6 non-interesting parameter dimensions. Requiring that the curve be continuous, by using de Rham's continuity condition 1, eliminates two more degrees of freedom. This leaves behind a four-dimensional space of unique fractal curves.

Of this four-dimensional space, one dimension has been explored with the Takagi curves. A second dimension is explored with the Cesàro curves, and a third with the Koch/Peano curves.

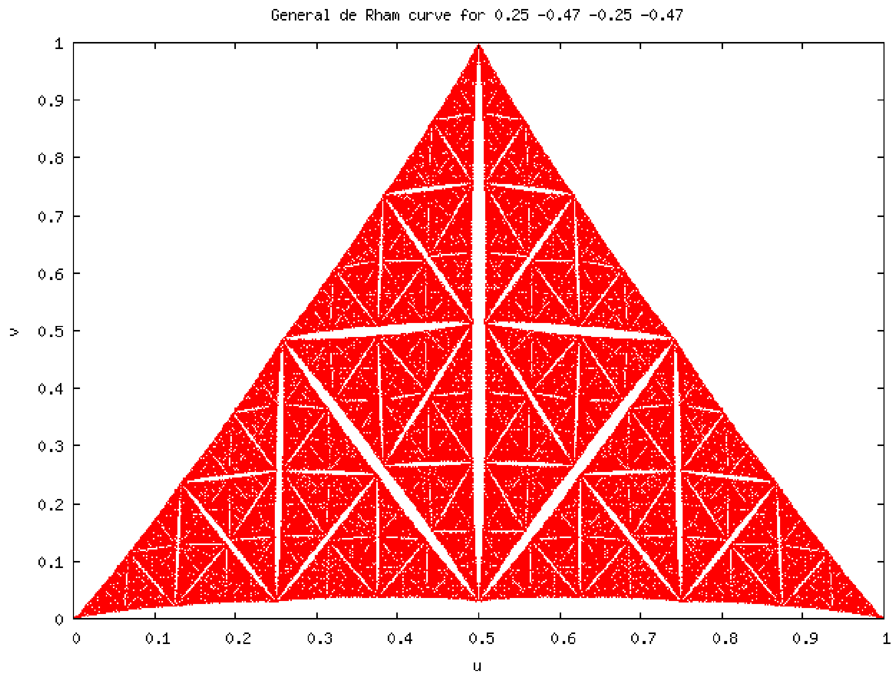
The general form may be narrowed as follows. Let d_0 have the fixed point p_0 located at the origin $(u,v) = (0,0)$. This implies that $a = d = 0$. Next let d_1 have the fixed point p_1 at $(u,v) = (1,0)$. This implies that $j = 1-h$ and $m = -l$. Finally, impose the de Rham condition for the continuity of the curve, namely that $d_0(p_1) = d_1(p_0)$. This implies that $h = b$ and $m = e$. Changing symbols, the general form with the endpoints fixed may be written as

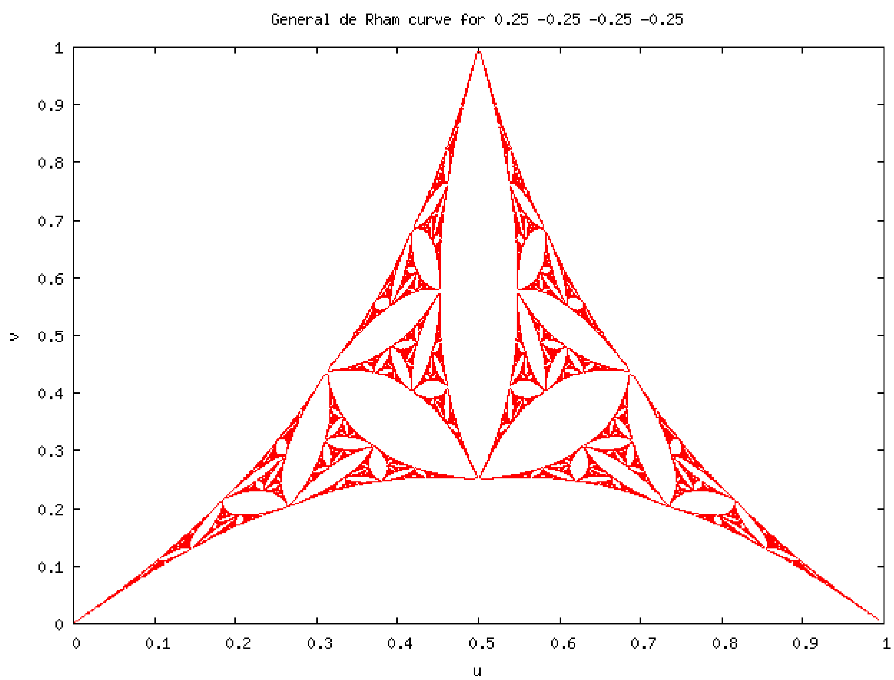
$$d_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \delta \\ 0 & \beta & \varepsilon \end{pmatrix} \text{ and } d_1 = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1-\alpha & \zeta \\ \beta & -\beta & \eta \end{pmatrix}$$

The half-way point of this curve is located at $1/2 = 0.100\ldots = 0.011\ldots = d_1 d_0 d_0 \ldots = d_0 d_1 d_1 \ldots$ which can be seen to be $(u,v) = (\alpha, \beta)$. Using this last result to fix the location of the half-way point, what remains is a four-parameter family of linear planar fractal curves. Counting shearing of the space, then one has a five-parameter family.

2 Gallery

The following is a collection of images exploring this space of linear affine curves. Each picture is labelled with four numbers. These numbers are δ, ε, ζ and η. In all cases, α and β have been fixed to $\alpha = 0.5$ and $\beta = 1.0$. The vertical and horizontal axes are properly label, thus all curves start at $(u,v) = (0.0)$ and end at $(u,v) = (1,0)$.


General de Rham curve for 0.25 -0.47 -0.25 -0.47
v
u
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1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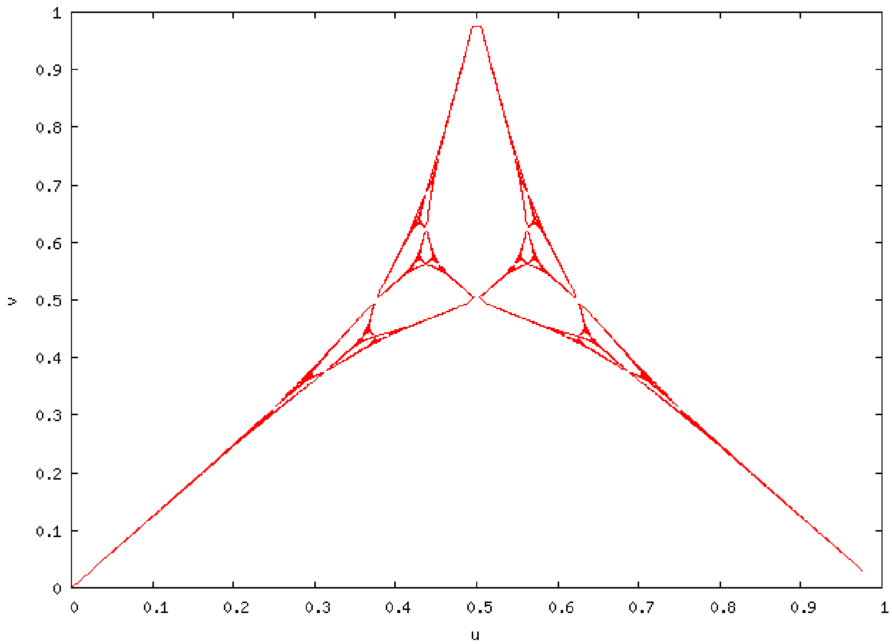
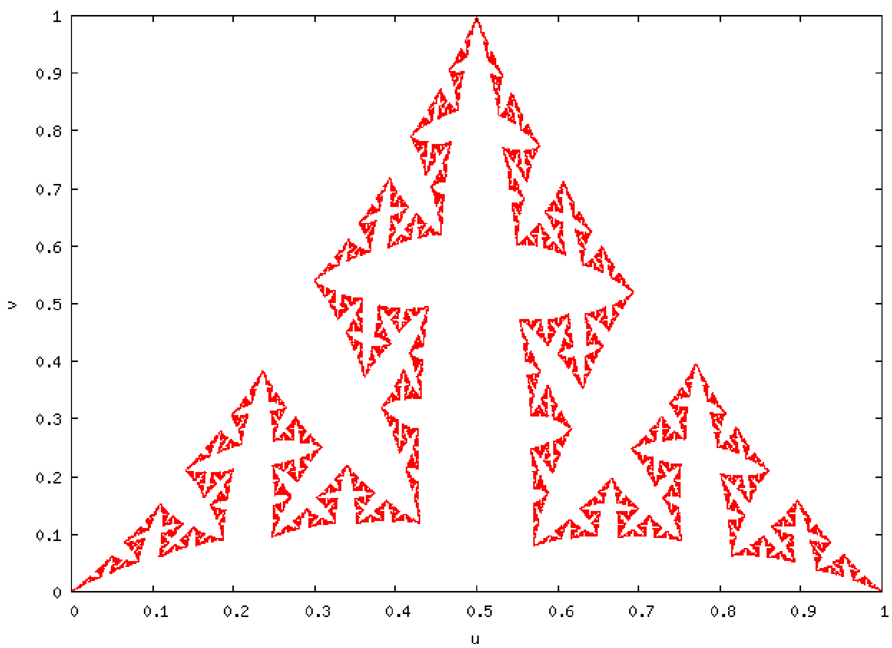
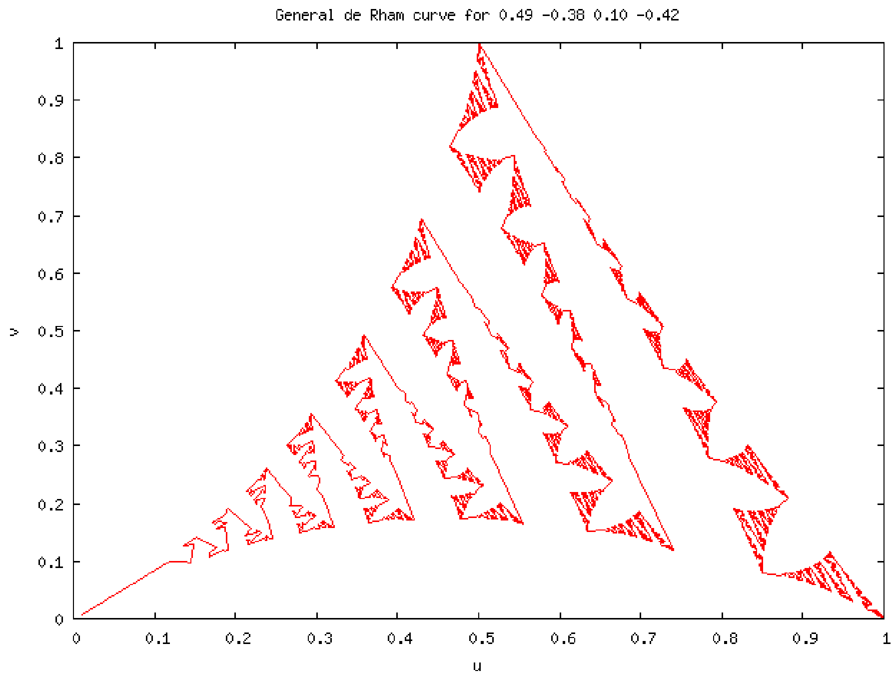
v
u
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

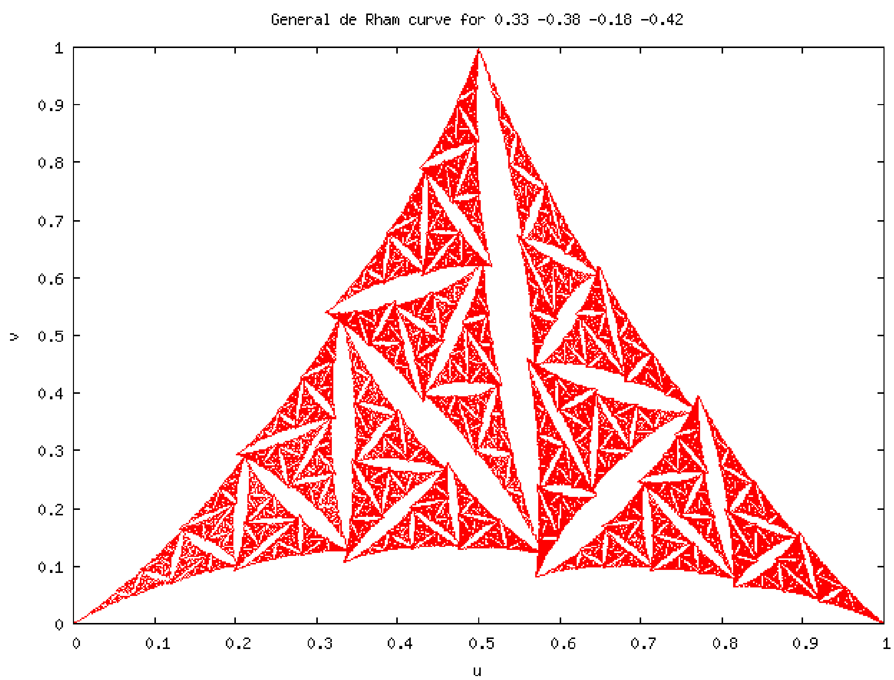
General de Rham curve for 0.25 0.00 -0.25 0.00

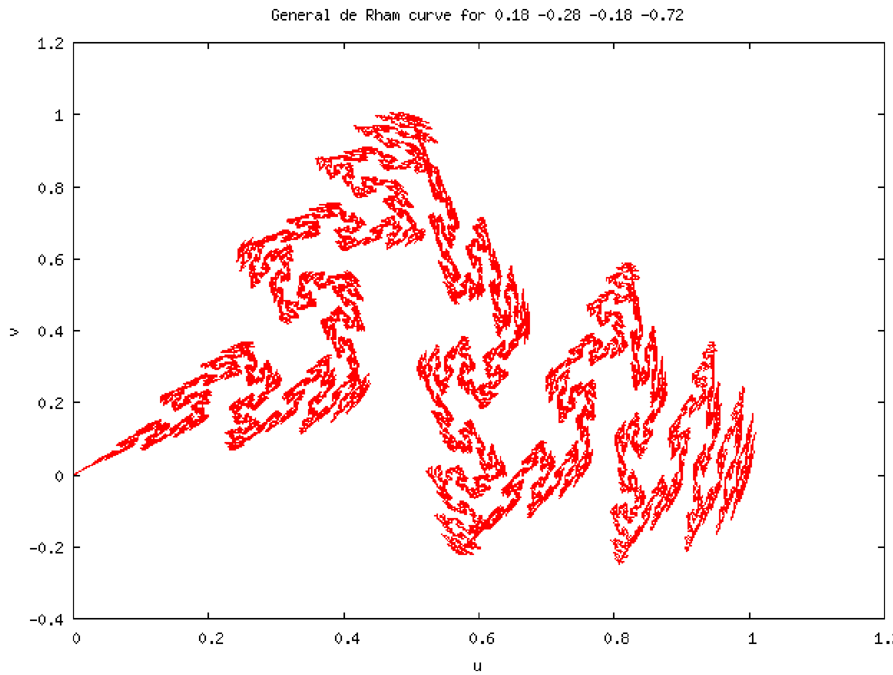


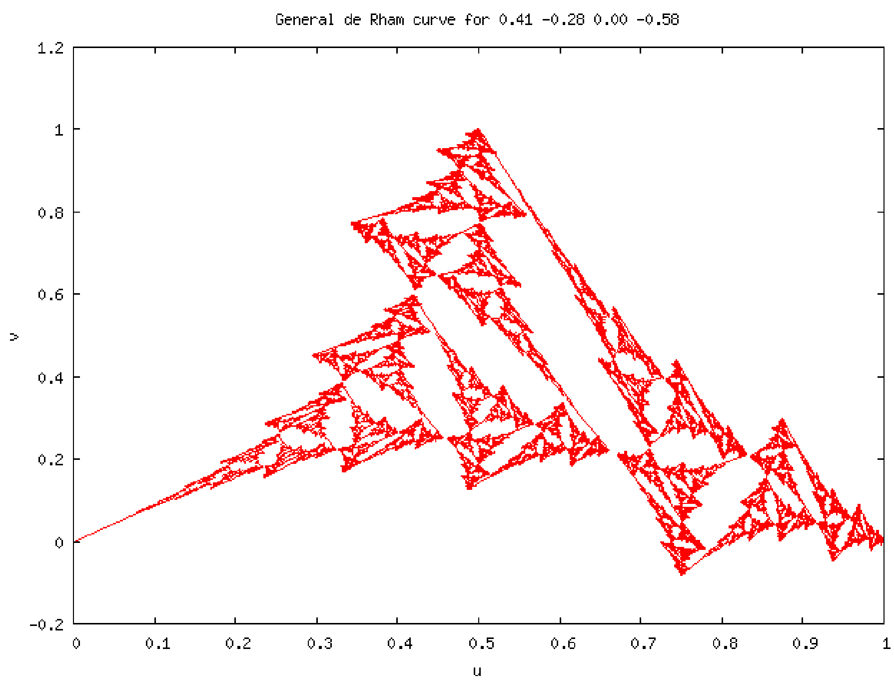
General de Rham curve for 0.18 -0.38 -0.18 -0.42

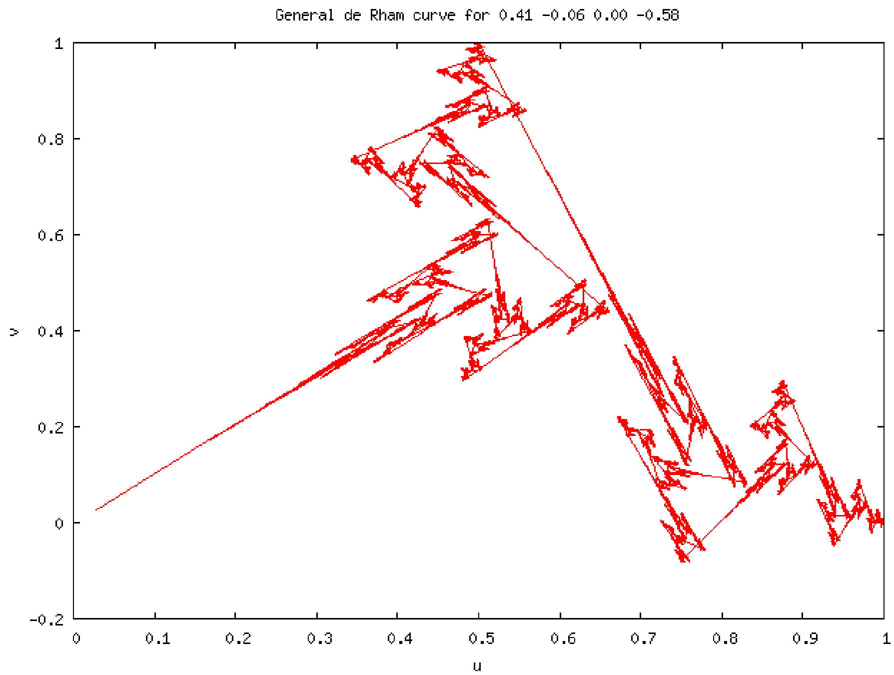



General de Rham curve for 0.49 -0.38 0.10 -0.42
v
u


General de Rham curve for 0.33 -0.38 -0.18 -0.42
v
u


General de Rham curve for 0.18 -0.28 -0.18 -0.72
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
v
0
0.2
0.4
0.6
0.8
1
1.2
u


General de Rham curve for 0.41 -0.28 0.00 -0.58
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
v
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
u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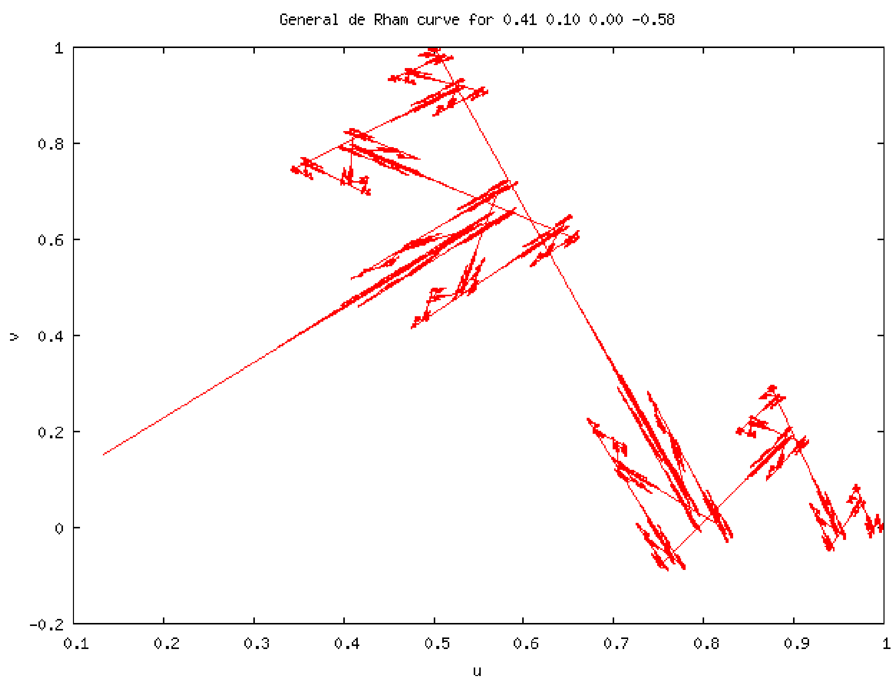
v
u
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


General de Rham curve for 0.41 0.10 0.00 -0.58
v
u
1
0.8
0.6
0.4
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0.2
0.3
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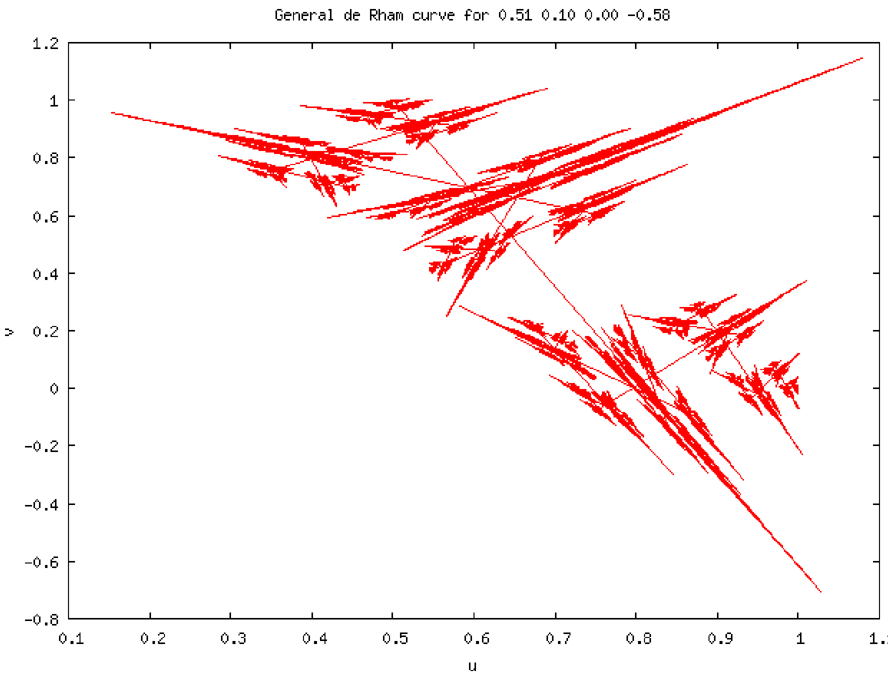
0.7
0.8
0.9
1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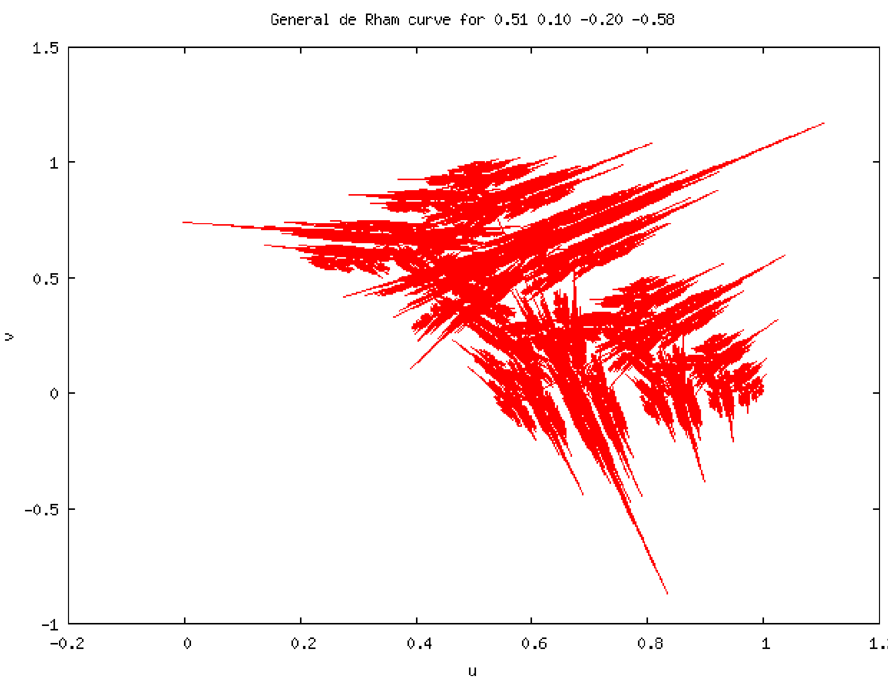
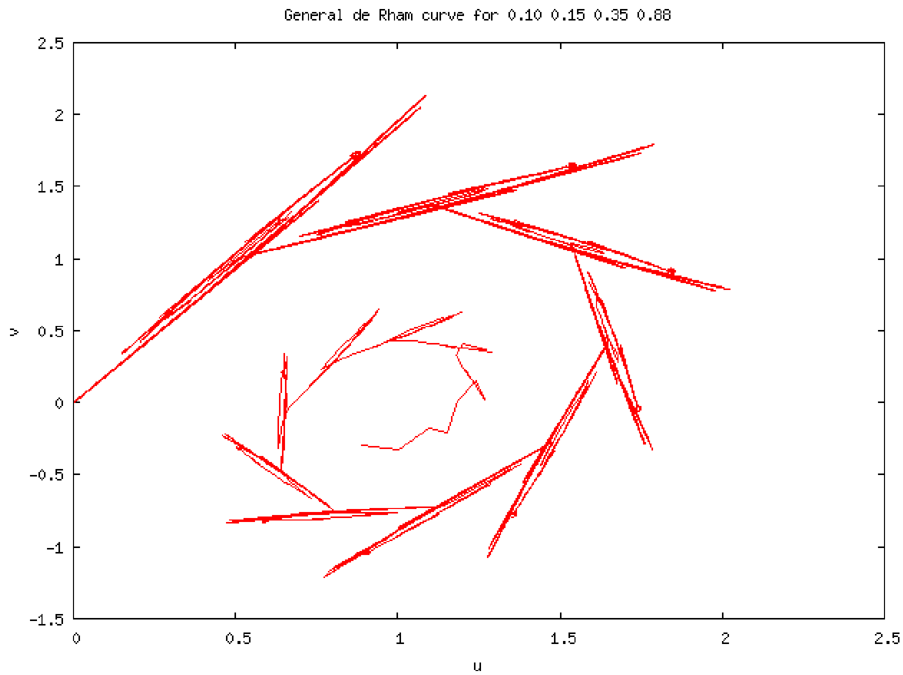
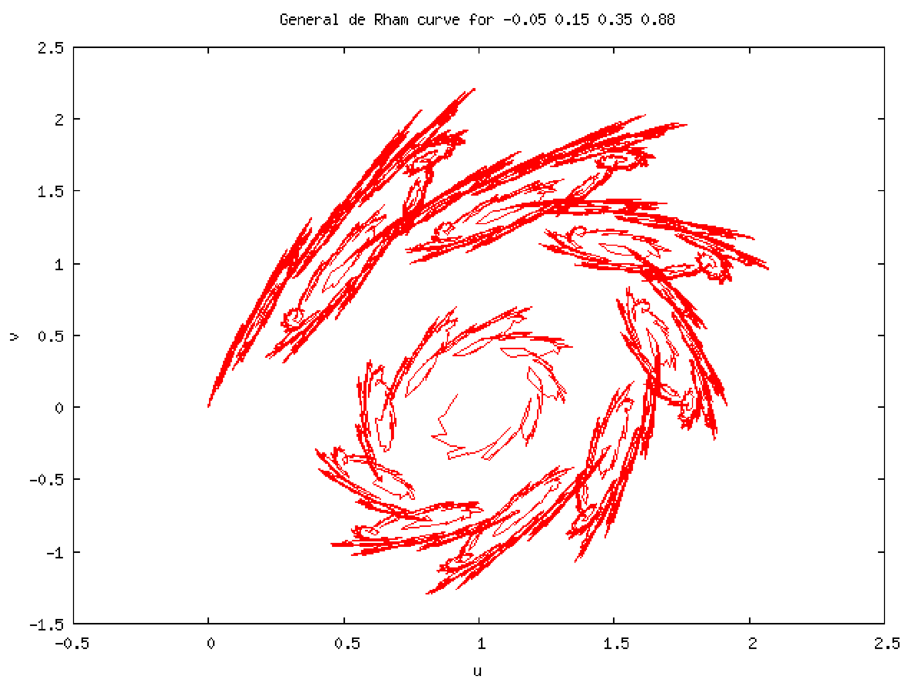
v
u
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1


General de Rham curve for 0.51 0.10 -0.20 -0.58
v
u
1.5
1
0.5
0
-0.5
-1
-0.2
0
0.2
0.4
0.6
0.8
1
1.2


General de Rham curve for 0.10 0.15 0.35 0.88
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
v
0
0.5
1
1.5
2
2.5
u


General de Rham curve for -0.05 0.15 0.35 0.88
2.5
2
1.5
1
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v
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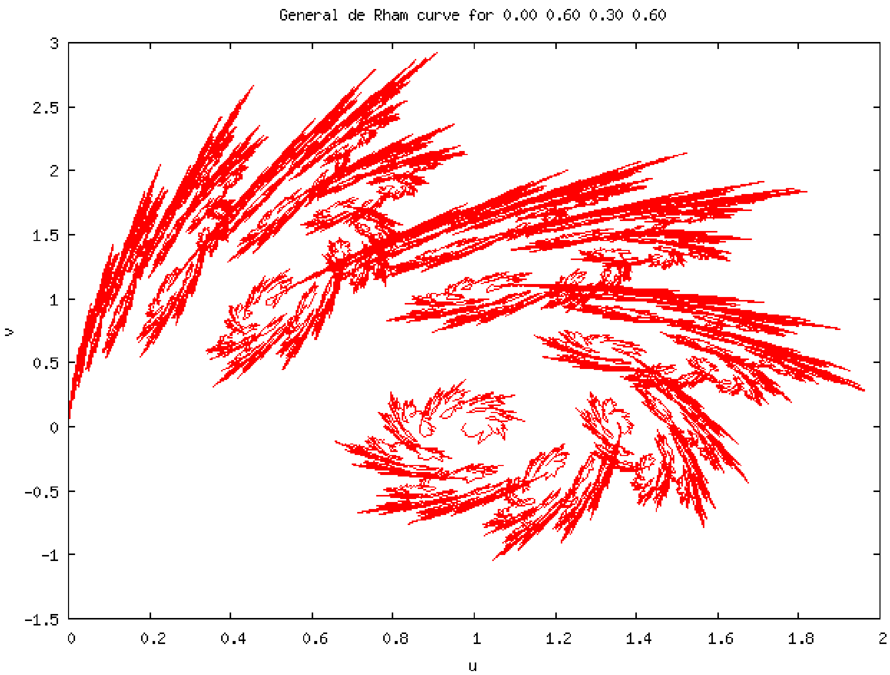
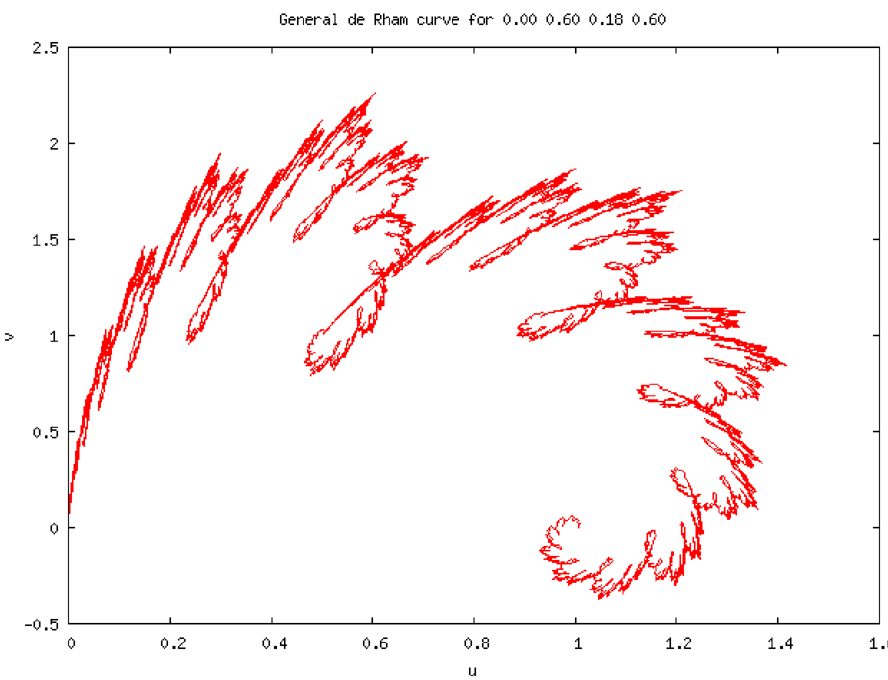
0.5
1
1.5
2
2.5
u


General de Rham curve for 0.00 0.60 0.30 0.60
v
u
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2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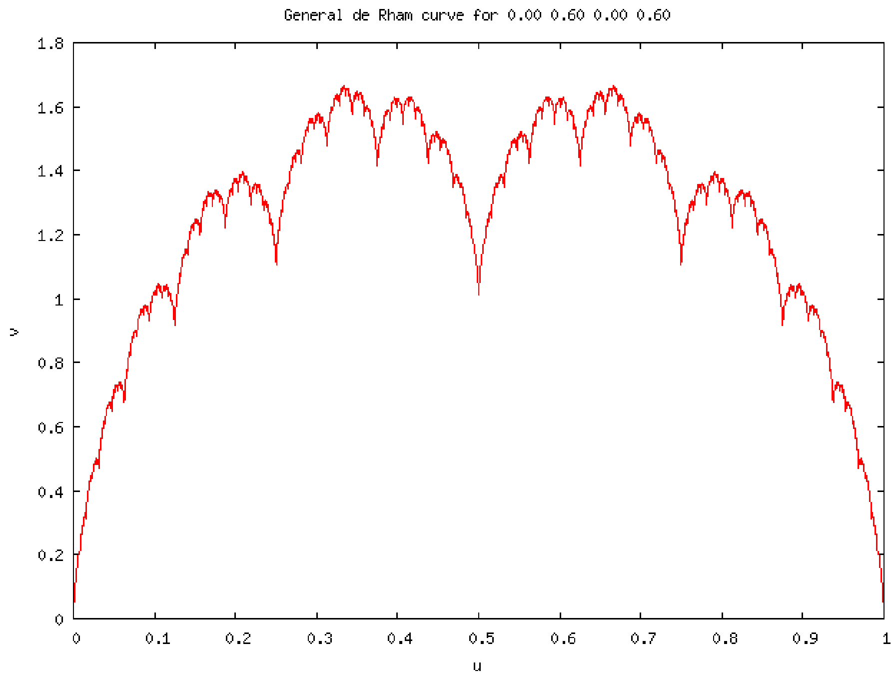
General de Rham curve for 0.00 0.60 0.00 0.60
v
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0.2
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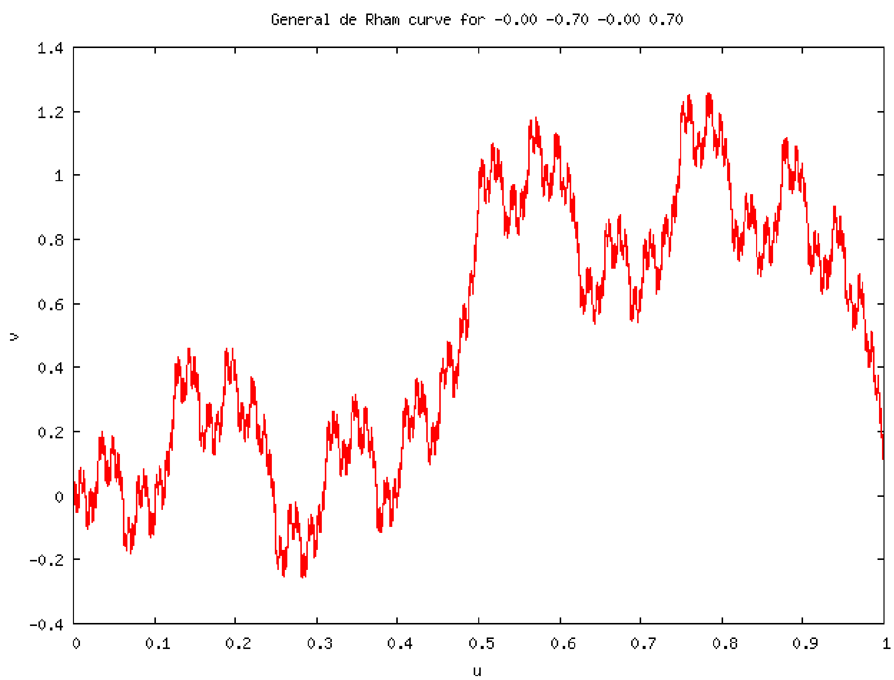
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v
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-0.4
-0.2
0
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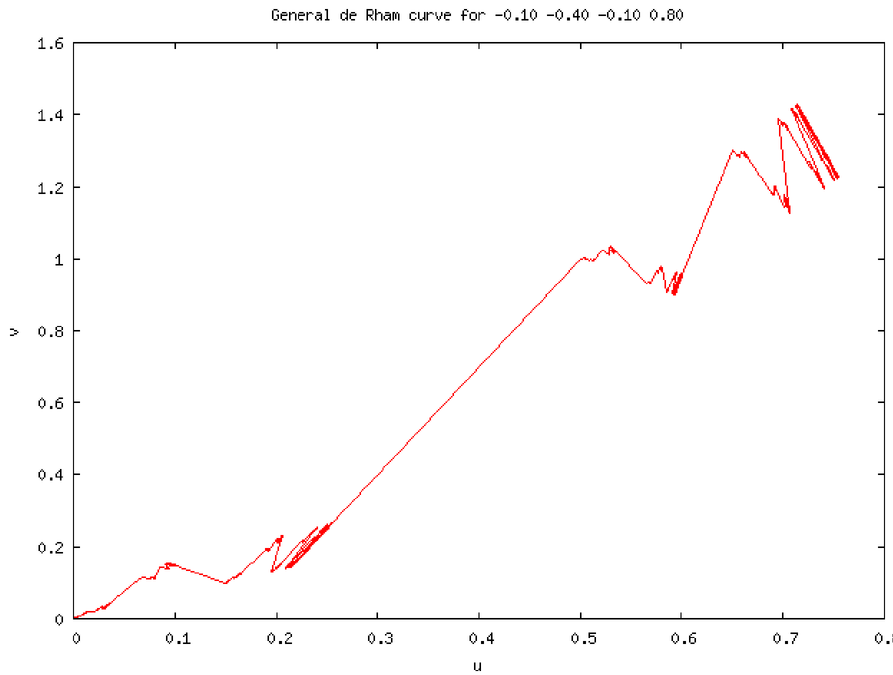
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v
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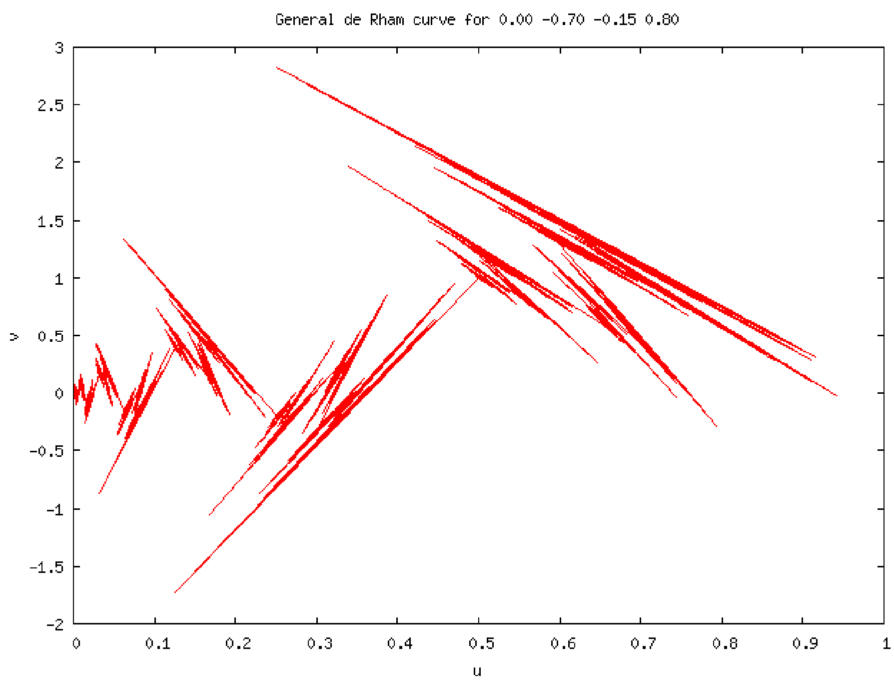
1.6
0.1
0.3
0.5
0.7
0.8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v
u
-2
-1.5
-1
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0
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1
1.5
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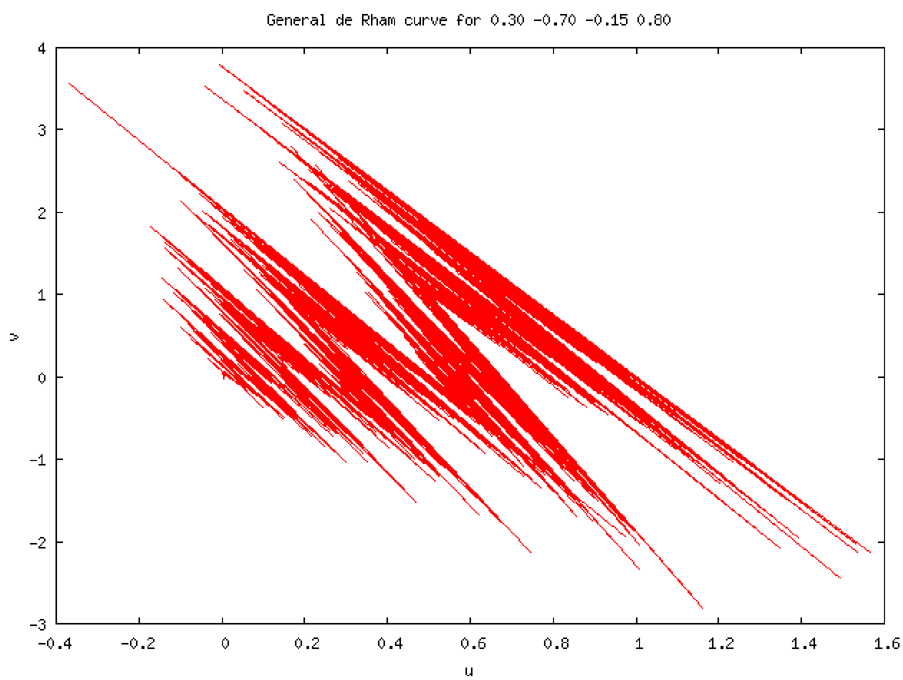
0.9
1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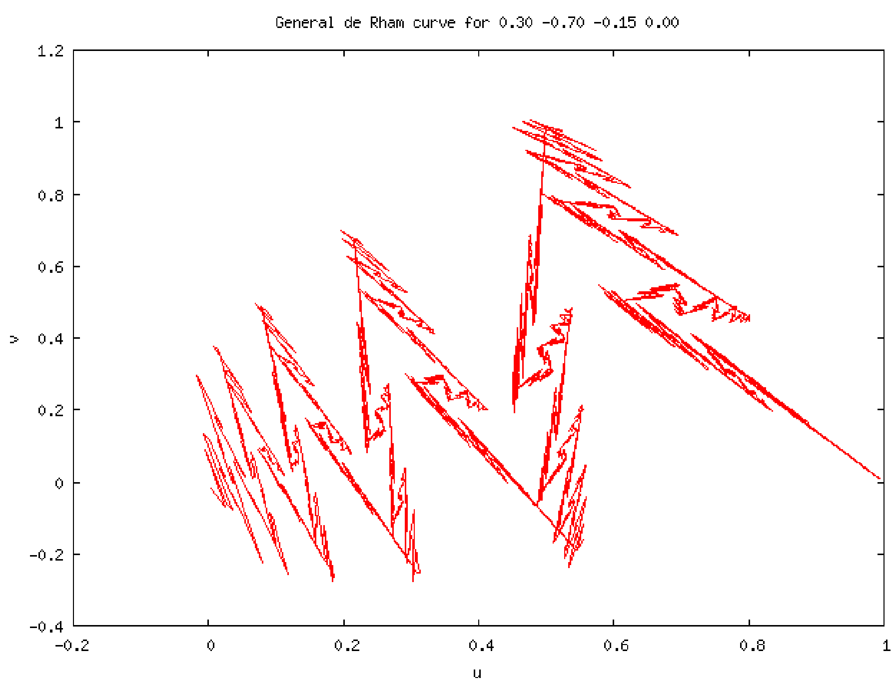
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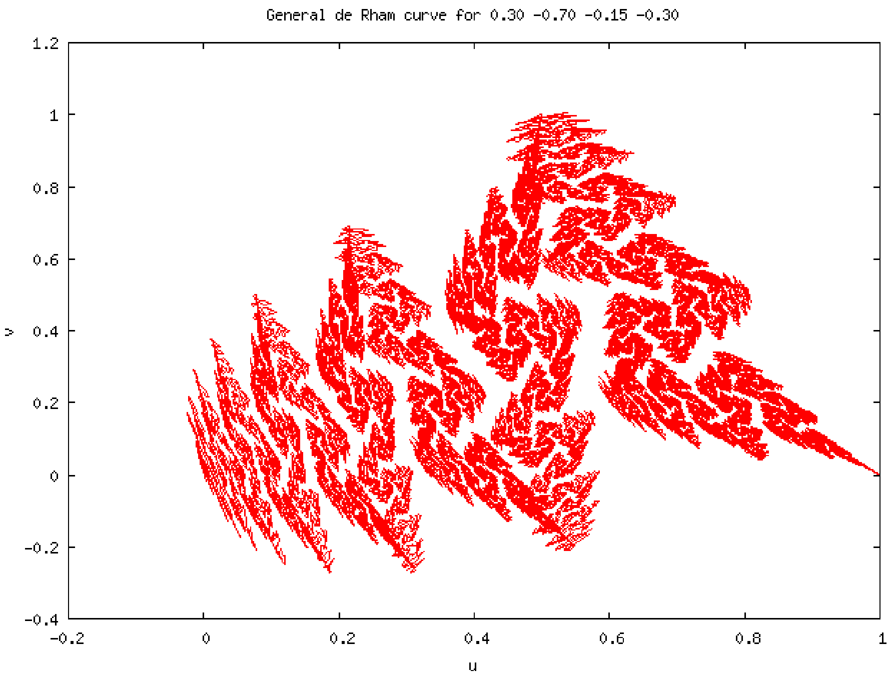
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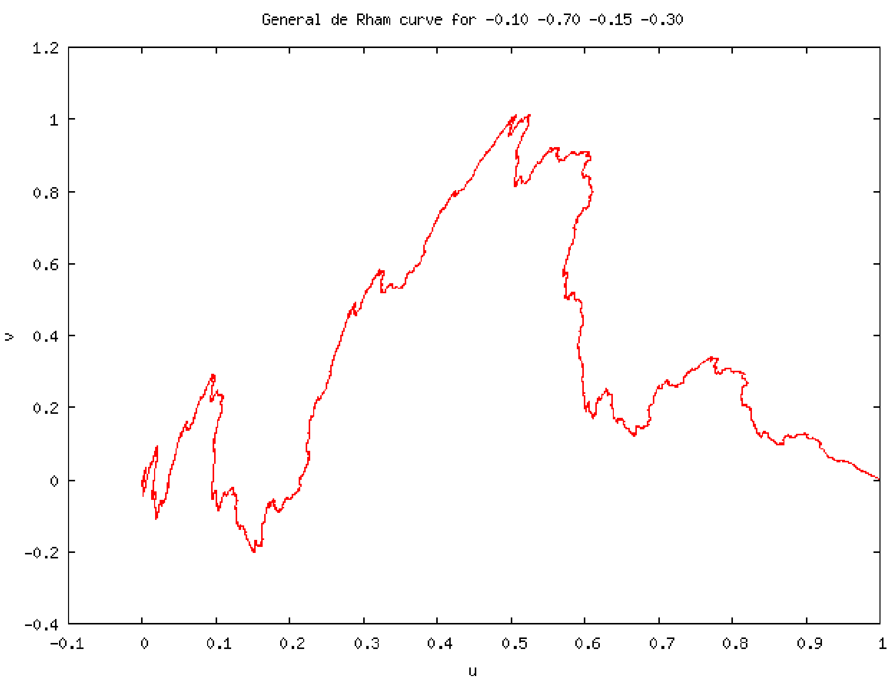
v
u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v
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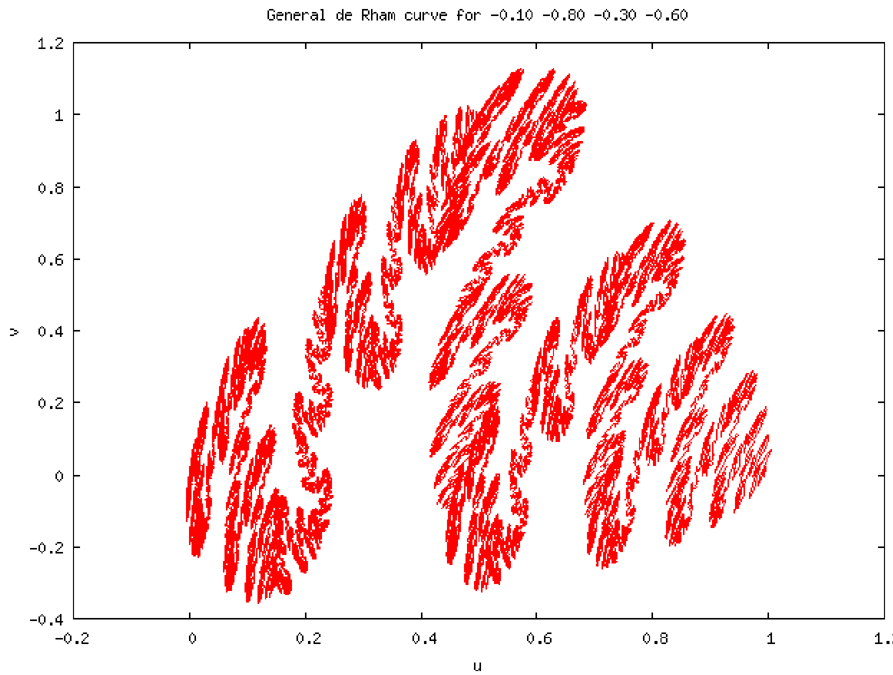
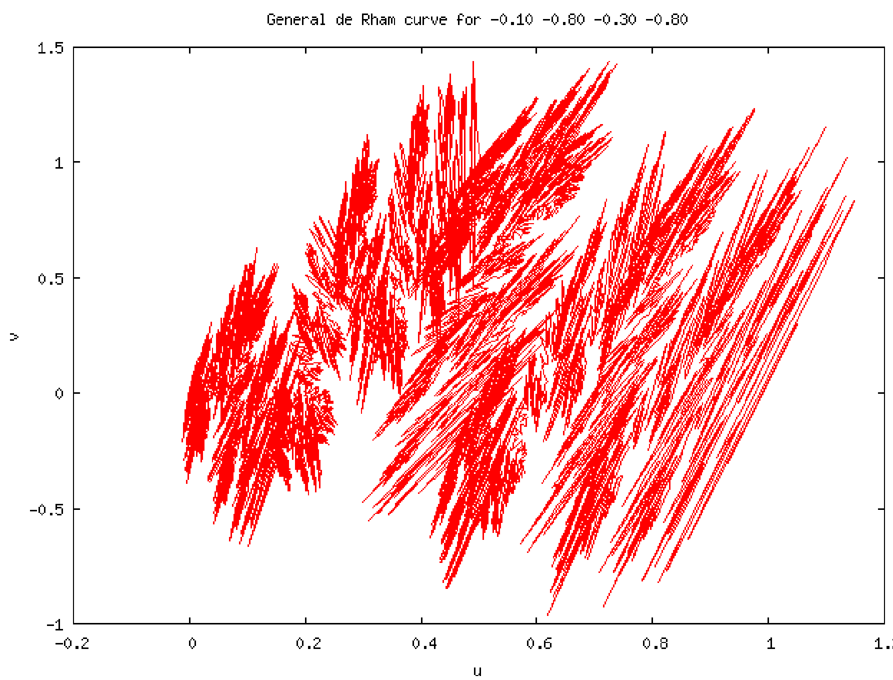
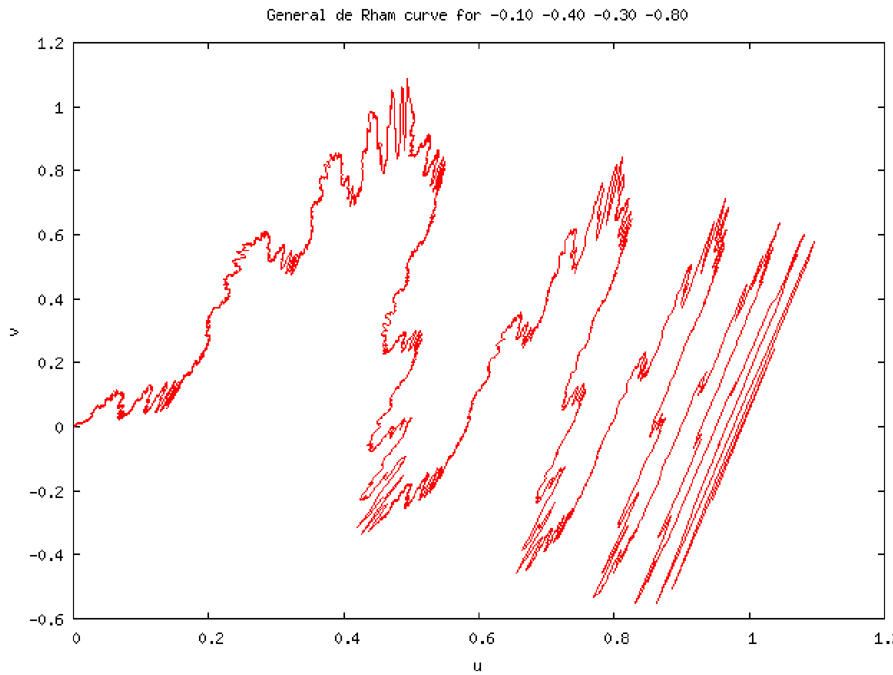
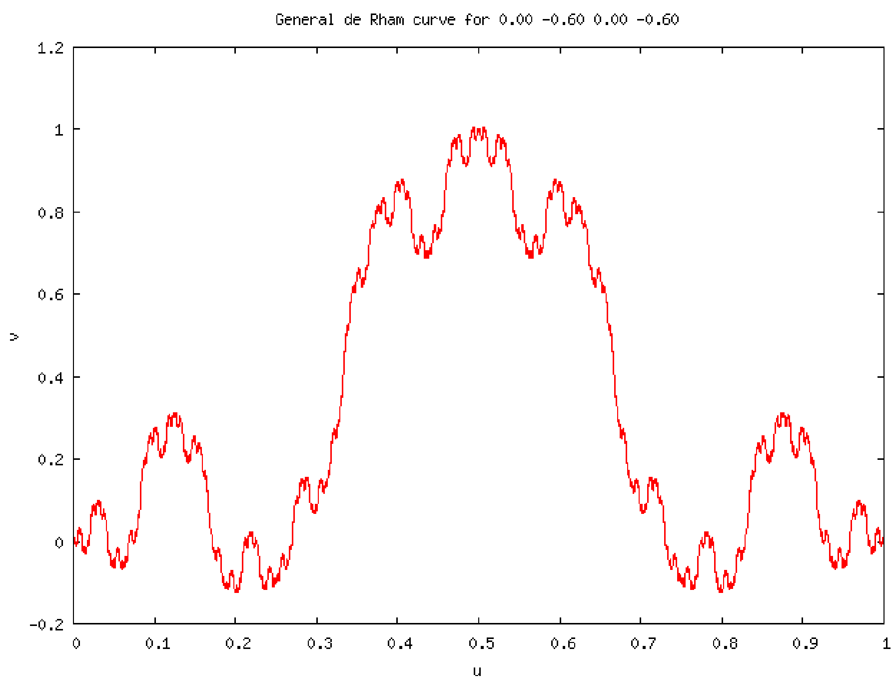
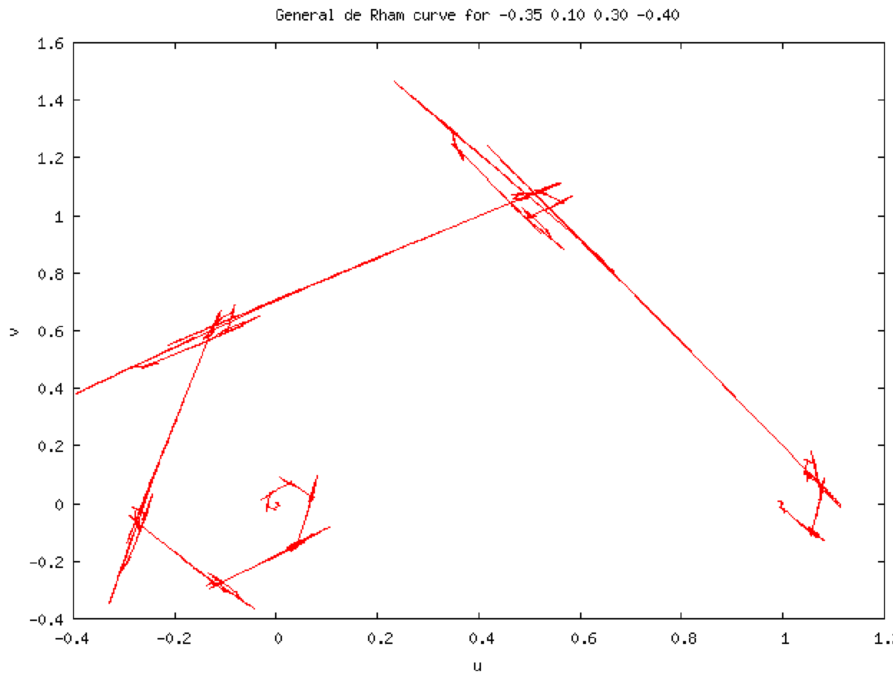
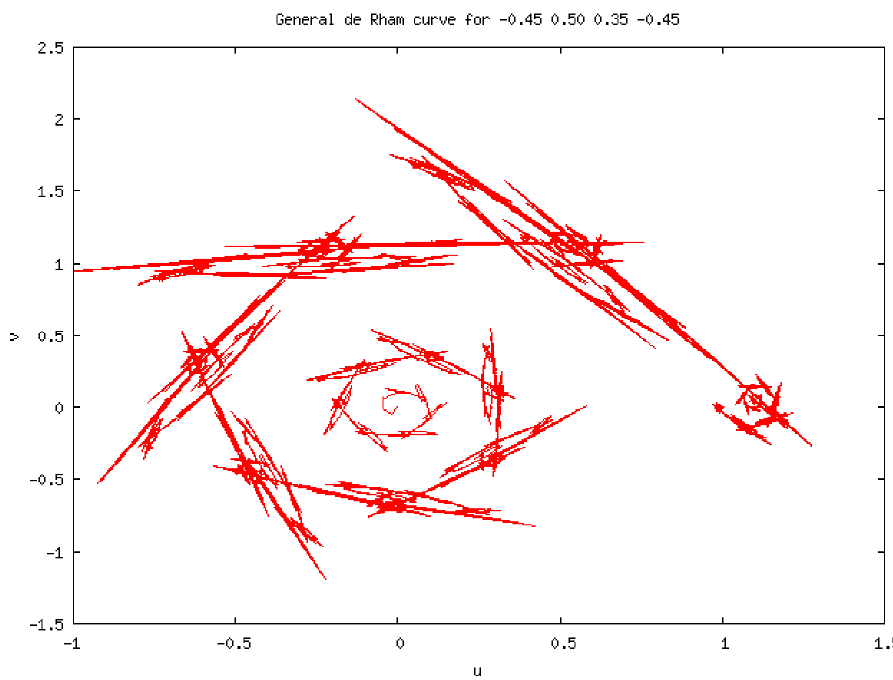

General de Rham curve for -0.10 -0.80 -0.30 -0.60
v
u

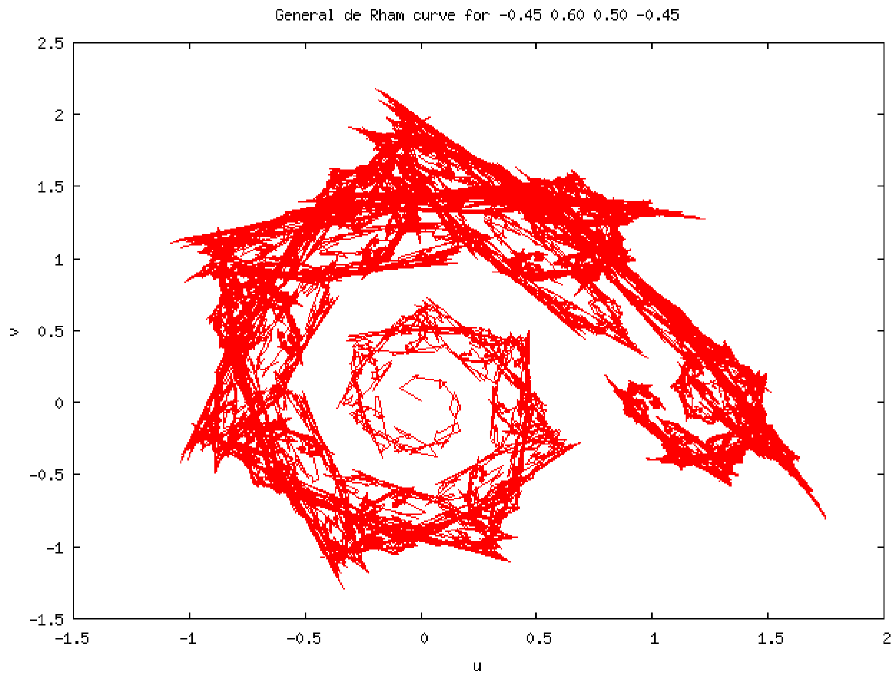

General de Rham curve for -0.10 -0.80 -0.30 -0.80
v
u

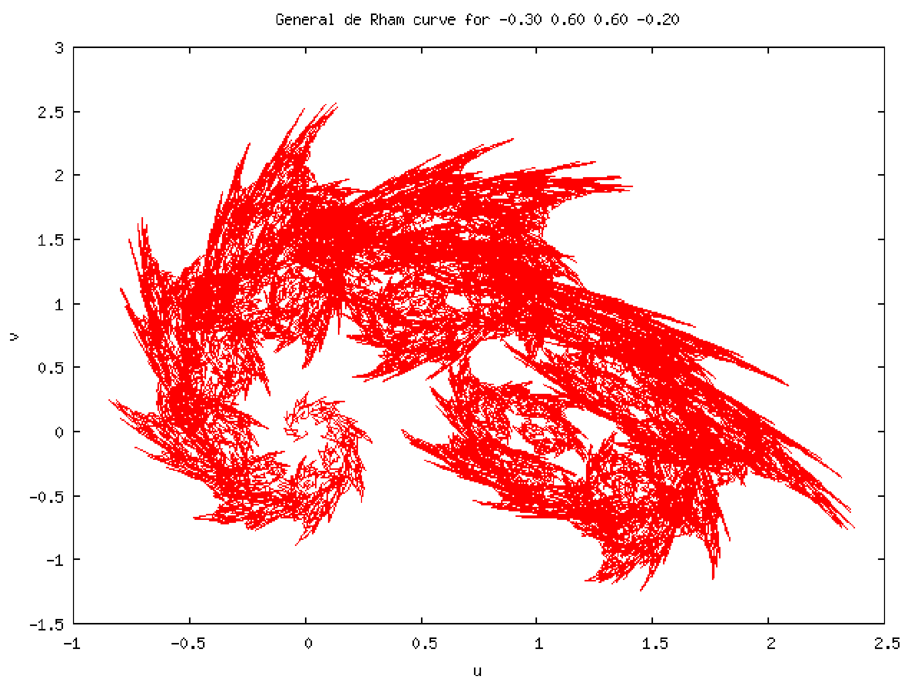

General de Rham curve for -0.10 -0.40 -0.30 -0.80
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
v
0
0.2
0.4
0.6
0.8
1
1.2
u

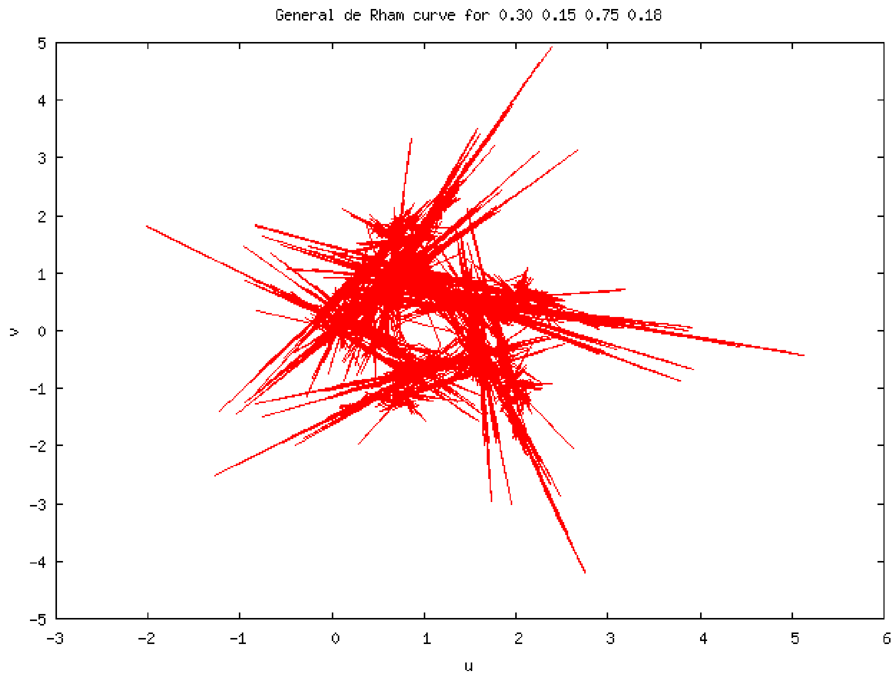

General de Rham curve for 0.00 -0.60 0.00 -0.60
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
v
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
u

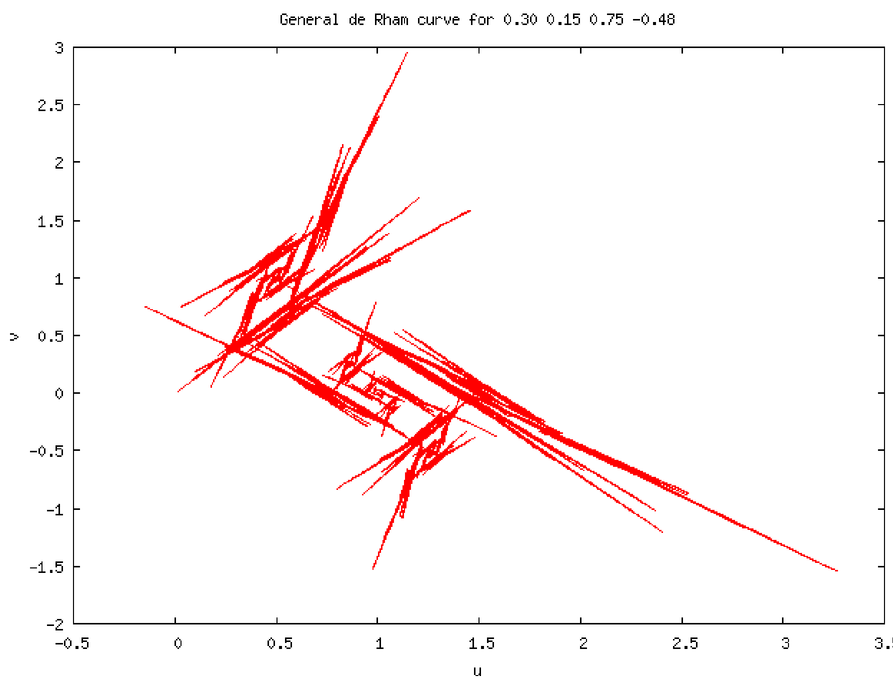

General de Rham curve for -0.35 0.10 0.30 -0.40
v
u
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
1.2

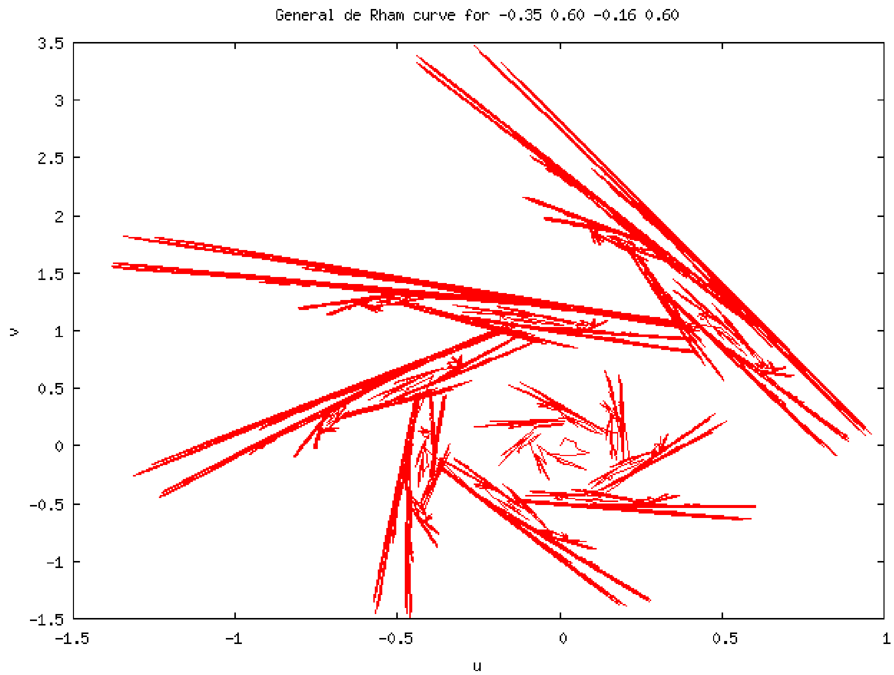

General de Rham curve for -0.45 0.50 0.35 -0.45
v
u
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1
1.5

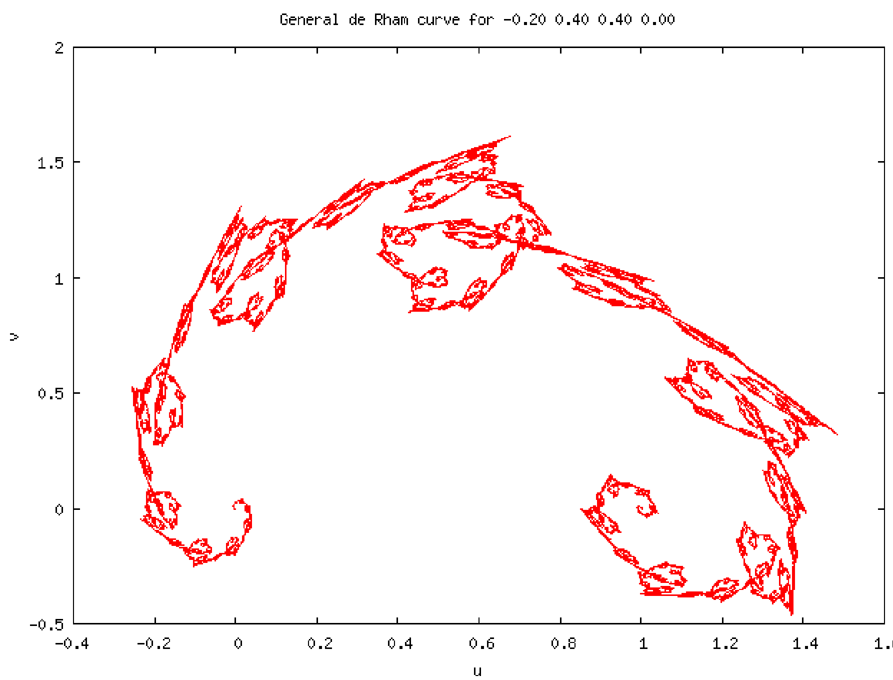

General de Rham curve for -0.45 0.60 0.50 -0.45
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
v
-1.5
-1
-0.5
0
0.5
1
1.5
2
u


General de Rham curve for -0.30 0.60 0.60 -0.20
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
v
-1
-0.5
0
0.5
1
1.5
2
2.5
u

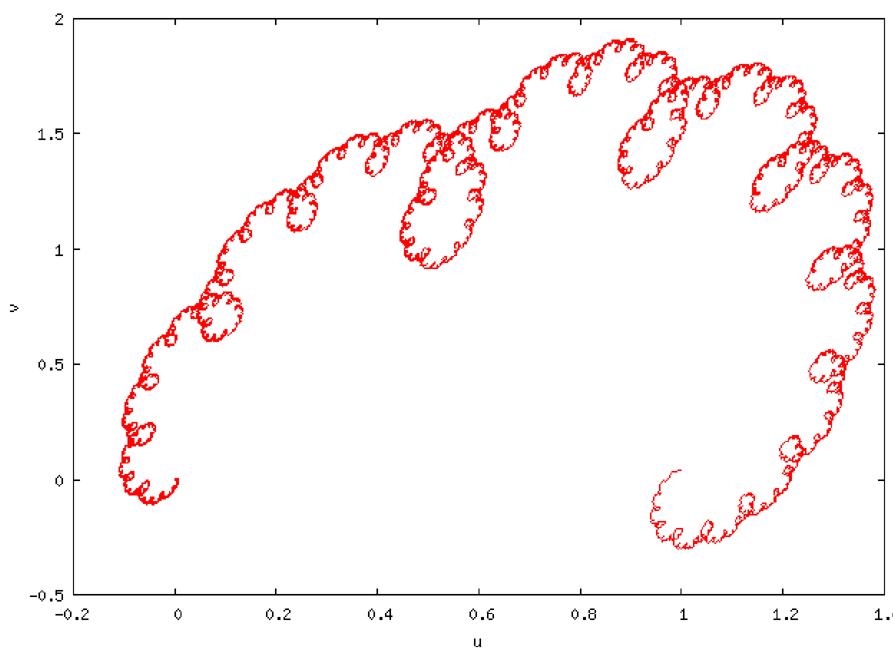

General de Rham curve for 0.30 0.15 0.75 0.18
u
v


General de Rham curve for 0.30 0.15 0.75 -0.48
u
v

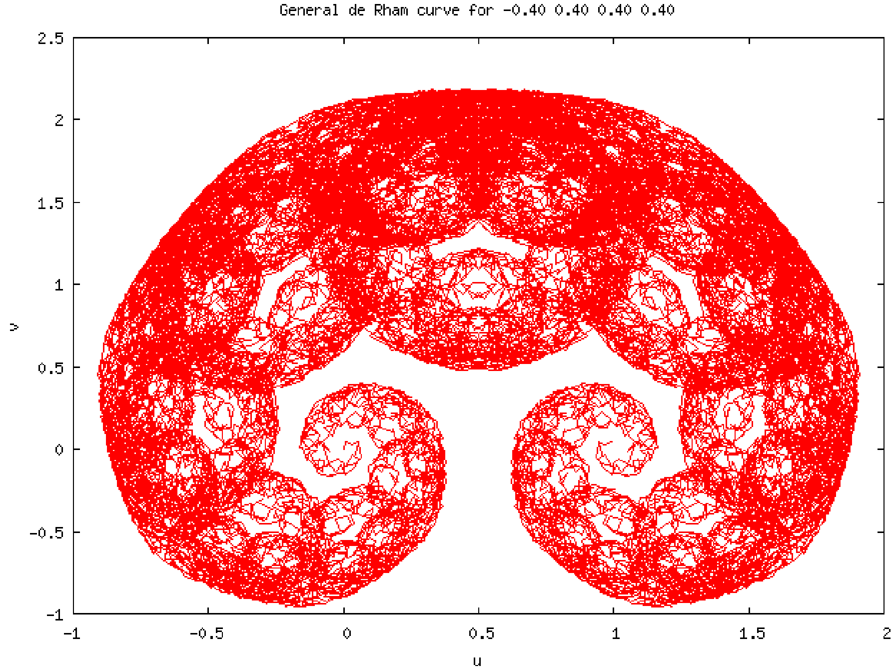

General de Rham curve for -0.35 0.60 -0.16 0.60
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
v
-1.5
-1
-0.5
0
0.5
1
u

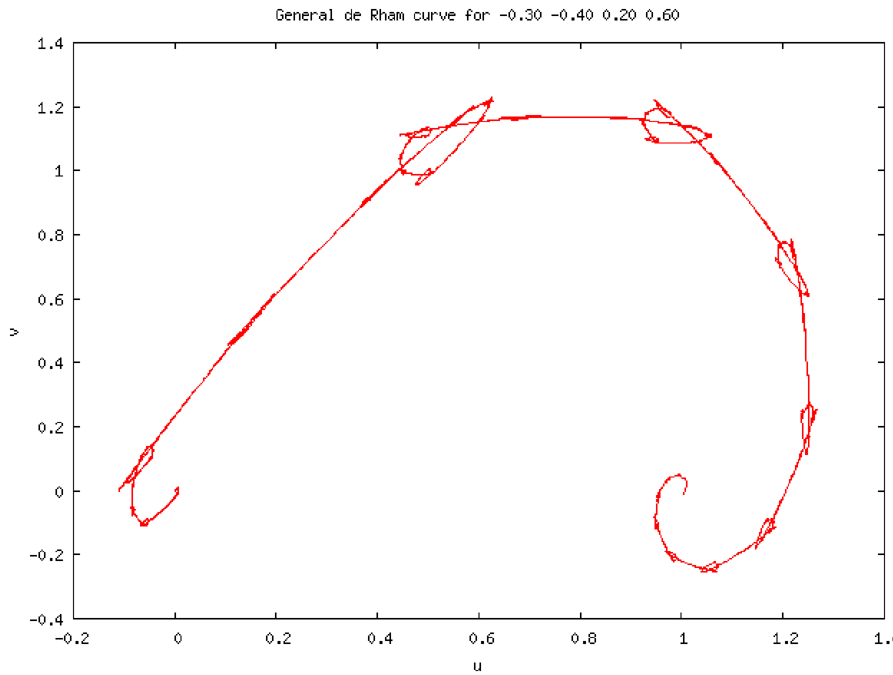

General de Rham curve for -0.20 0.40 0.40 0.00
2
1.5
1
0.5
0
-0.5
v
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
u

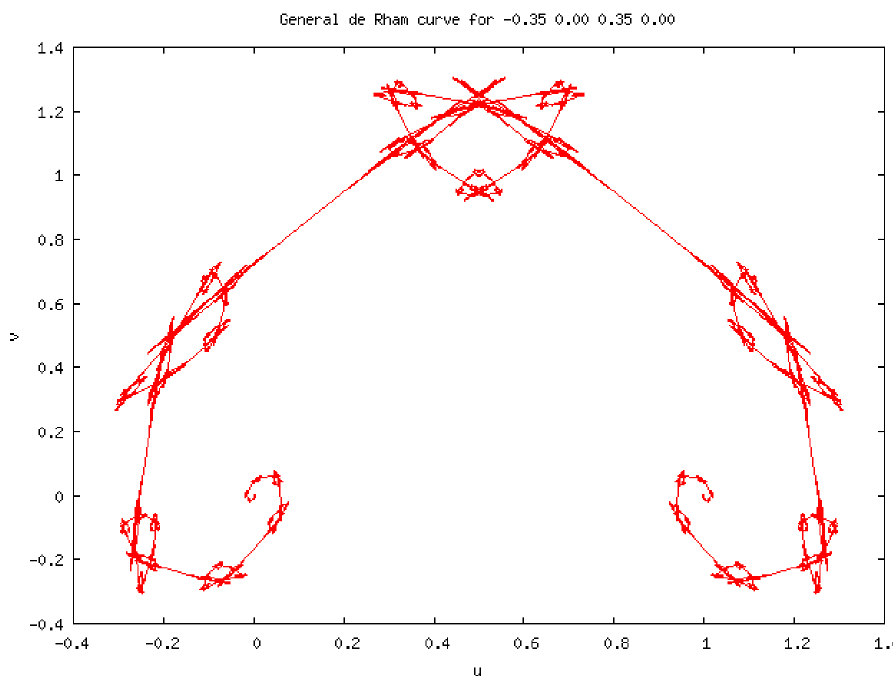
General de Rham curve for -0.15 0.15 0.15 0.85

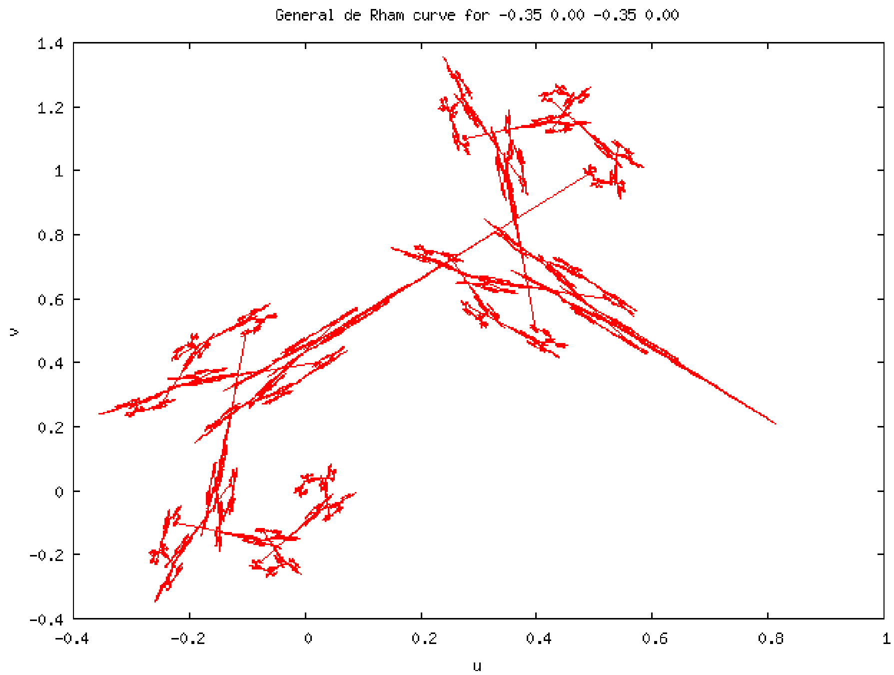


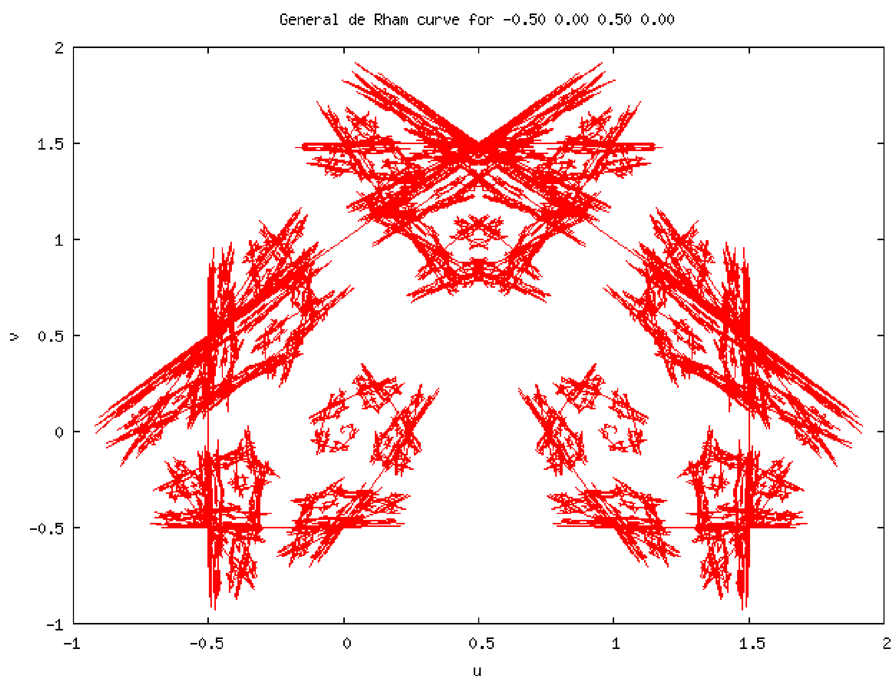
General de Rham curve for -0.40 0.40 0.40 0.40




General de Rham curve for -0.30 -0.40 0.20 0.60
v
u
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4


General de Rham curve for -0.35 0.00 0.35 0.00
v
u
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4


General de Rham curve for -0.35 0.00 -0.35 0.00
v
u
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


General de Rham curve for -0.50 0.00 0.50 0.00
v
u
2
1.5
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
1.5
2

3 Conclusions

Goolly! An open question: Can these images all be taken to be the projections of some general curve in some higher-dimensional space, projected down to two dimensions, along some axis?

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