

THE BERNOULLI OPERATOR

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ABSTRACT. This paper reviews a raft of related ideas surrounding the Bernoulli operator. The Bernoulli operator is the transfer operator or Frobenius-Perron operator of the Bernoulli map. The Bernoulli map is a simple map of the unit interval onto itself, which has the effect of discarding the leading binary digit of the binary expansion of a number upon every iteration. This map has been well studied in the literature, and much, if not all, of what is presented here is well-known.

If there is anything new here, then perhaps it is a representation of the Bernoulli map eigenfunctions in terms of the Takagi curve (or Blancmange curve). In particular, it is shown that the Hurwitz-zeta function basis for the continuous spectrum is just a linear combination of the Takagi curves, and vice-versa. I find this curious and important somehow: the Blancmange curve has an explicitly fractal self-similarity given by the dyadic monoid. The dyadic monoid is the monoid that describes the self-similarity of the infinite binary tree, and is a subset of the modular group $SL(2, \mathbb{Z})$.

This paper is part of a set of chapters that explore the relationship between the real numbers, the modular group, and fractals.

1. THE BERNOULLI OPERATOR

THIS IS A SET OF WORKING NOTES. Its somewhat loosely structured, sometimes messy, often presents topics in backwards order, and occasionally assumes familiarity with Linas' other writings and/or a general knowledge of various unspecified branches of mathematics. The intro hasn't been written yet; but if it was, it would mention the following:

The general layout is to move from a less formal, to a more rigorous, more formal presentation.

- Define the Bernoulli operator, demonstrate some of its eigenfunctions and eigenvalues. The presentation is a simplified variant of the material in [?], although generalized to p -adic definitions.
- Present the Hurwitz Zeta and fractal eigenfunctions; show that these are just linear combinations of one-another, and span the same space. Notable is the fact that the fractal eigenfunctions are not differentiable on the rational numbers, whereas the Hurwitz Zeta eigenvalues are classically differentiable almost everywhere.
- Present a more abstract, but more firm/well-founded foundation for the earlier results. The more abstract formulation is in terms of the product space $\Omega = 2^\omega$ and the product topology on it. This foundation can then lead to deeper results, as well as shed light on some of the more confusing/paradoxical results that arise in the informal treatment.

Some of the sections, including the section on orthogonality and completeness, are awkwardly presented. The topic is subtle, the notation is not elegant. In a certain sense, this paper illustrates all the wrong ways in which to present the notions of completeness with regards to a Hilbert space. The notions of what is fractal, and what is differentiable, and

what is an operator, and what is an eigenvalue, is muddled as a result. The journey, however faulty, is still educational, though.

2. INTRODUCTION

The method of the transfer operator was introduced by David Ruelle[need ref] as a powerful mechanism for studying the nature of iterated maps. The transfer operator, sometimes called the Frobenius-Perron operator, or the Ruelle-Frobenius-Perron operator, provides a means to escape the narrow confines of point-set topology when considering an iterated function, and instead explore the function using wildly different topologies. In its most concrete form, it is a linear operator acting on a Banach space of functions. However, the structure and the properties of the operator depend very much on which space of functions one considers. In the following, the spaces of polynomial functions of the real numbers, as well as the space of square-integrable functions will be considered. More broadly, one may also consider the transfer operator acting on other spaces, such as the p -adic numbers, endowed with unusual topologies: in particular, with the so called “product topology” whose basis are the cylinder sets. In this topology, the transfer operator can be made to resemble the lattice models of theoretical physics, such as the Ising model, thus offering additional means of gaining insight.[?]

It is easiest to begin with the concrete definition. Consider a function $g : [0, 1] \rightarrow [0, 1]$, that is, a function mapping the unit interval of the real number line to itself. Upon iteration, the function may have fixed points or orbits of points. These orbits may be attractors or repellers, or may be neutral saddle points. The action of g may be ergodic or chaotic, strong-mixing or merely topologically mixing. In any case, the language used to discuss g is inherently based on either the point-set topology of the unit interval, or the “natural” topology on the unit interval, the topology of open sets.

A shift in perspective may be gained not by considering how g acts on points or open sets, but instead by considering how g acts on distributions on the unit interval. Intuitively, one might consider a dusting of points on the unit interval, with a local density given by $\rho(x)$ at point $x \in [0, 1]$, and then consider how this dusting or density evolves upon iteration by g . This verbal description may be given form as

$$(2.1) \quad \rho'(y) = \int_0^1 \delta(y - g(x)) \rho(x) dx$$

where $\rho'(y)$ is the new density at point $y = g(x)$ and δ is the Dirac delta function.

In this viewpoint, g becomes an operator that maps densities ρ to other densities ρ' , or notationally,

$$\mathcal{L}_g \rho = \rho'$$

The operator \mathcal{L}_g is called the transfer operator or the Ruelle-Frobenius-Perron operator. It is not hard to see that it is a linear operator, in that

$$\mathcal{L}_g(a\rho_1 + b\rho_2) = a\mathcal{L}_g\rho_1 + b\mathcal{L}_g\rho_2$$

for constants a, b and densities ρ_1, ρ_2 .

When the function g is differentiable, and doesn't have a vanishing derivative, the integral formulation of the transfer operator above can be rephrased in a more convenient form, as

$$(2.2) \quad [\mathcal{L}_g \rho](y) = \sum_{x: y=g(x)} \frac{\rho(x)}{|dg(x)/dx|}$$

where the sum is presumed to extend over at most a countable number of points. If these conditions do not hold, a transfer operator can still be defined, although more care must be taken in its definition.

One generalization should be immediately apparent: although the word “density” implies that ρ is a smooth map from the unit interval to the non-negative reals, no such requirement need to be enforced: ρ may be a map from the unit interval to any ring R , and it need not be smooth, differentiable or even continuous. This generalization gives a very rich structure to \mathcal{L}_g : the precise form of \mathcal{L}_g will take will depend very strongly on R , whether its the reals \mathbb{R} , the complex numbers \mathbb{C} , or some other field or ring. It will also depend strongly on whether one restricts oneself to smooth functions, continuous functions, square-integrable functions, or some other function space. An adequate study requires reference to the specific topology that the function space is endowed with; many different topologies may be considered. That is, in general, one must consider \mathcal{L}_B to be an operator acting on a topological space endowed with multiplication and addition, that is, a topological vector space. A precise definition of the transfer operator is given in the next section, as being the pushforward onset of measurable, sigma-additive functions.

The structure of \mathcal{L}_g also depends on the topology applied to the unit interval. Besides the natural topology on the real number line, the unit interval can be given several other topologies. The most important of these is the Cantor set topology, or the p -adic topology. Here, one considers the unit interval $[0, 1]$ to consist of the set of strings

$$\Omega = \left\{ \sigma = (\sigma_0, \sigma_1, \sigma_2, \dots) : \sigma_k \in \{0, 1, \dots, p-1\}, x = \sum_{k=0}^{\infty} \sigma_k p^{-(k+1)}, x \in [0, 1] \right\}$$

Intuitively, this set is simply the set of all the digits of a base- p expansion of the real numbers $x \in [0, 1]$. The connection with physics comes from the realization that this set can be understood to be the collection of all field configurations of a one-dimensional, one-sided lattice, where each lattice location can take on one of p values. Such lattices are commonly given the product topology, where the open sets are the cylinder sets consisting of substrings of sequences of letters. The topology also has a natural measure, derived from the length of letter sequences. Aside from the p -adic expansion above, one also has the continued fraction expansion, where one considers the sequence of integers making up the continued fraction

$$x = [0; \sigma_1, \sigma_2, \sigma_3, \dots] = \frac{1}{\sigma_1 + \frac{1}{\sigma_2 + \frac{1}{\sigma_3 + \dots}}}$$

where each σ_k is a positive integer; the entire sequence again be given a product topology, although the measure is constructed differently.

In most of what follows, the topological rings that will be considered will be very concrete: these will be the Banach spaces of polynomial functions, and of square-integrable functions. Some exploration of the lattice-model topology is explored in [?]. Since \mathcal{L}_B is a linear operator, the primary focus of study is to characterize it along traditional lines: find its representations, find the eigenvectors and eigenspaces associated with each representation, discuss any symmetries and pertinent isomorphisms these spaces might have.

The eigenfunctions of \mathcal{L}_g are called Ruelle resonances [need citation], elaborate.

3. TRANSFER OPERATOR OF THE BERNOULLI MAP

The Bernoulli map is an exactly solvable example of deterministic chaos. A presentation of this map, its associated transfer operator and its solution in terms of polynomial

eigenfunctions is given by Driebe[?]. This and the next section recaps those results using a simplified development and simpler tools. The simplified development enables the discussion of more complex scenarios, given in later sections.

The Bernoulli map is given by

$$(3.1) \quad b(x) = 2x - \lfloor 2x \rfloor$$

where $\lfloor x \rfloor$ denotes the integer part of x . The map can be intuitively thought of as popping the leading digit off of the binary or 2-adic expansion of x . This map has a positive Lyapunov exponent and is highly chaotic, as, in a certain sense, one can say that the digits of the binary expansion of some 'arbitrary' number are unpredictable, and that the orbits of two close-by numbers will eventually become 'uncorellated' (after suitably defining what we mean by 'arbitrary' and 'unpredictable'). Closely related is the p -adic map, given by

$$(3.2) \quad a(x) = px - \lfloor px \rfloor$$

for p an integer. As above, this map has the effect of popping off the leading digit of the base- p expansion of x . In the same vein, one may also consider the Gauss map

$$(3.3) \quad g(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$$

which has the effect of lopping of the leading digit of the continued fraction expansion of x . What these three maps have in common is that they deal with different (and, in a certain sense, inequivalent) representations of the continuum of real numbers. Each map is chaotic, but in a different way.

The Ruelle-Frobenius-Perron operator or transfer operator of the Bernoulli map is given by

$$(3.4) \quad [\mathcal{L}_B f](x) = \frac{1}{2} \left[f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right) \right]$$

which follows directly from equation 2.2. Similarly, the transfer operator for the general p -adic map is

$$(3.5) \quad [\mathcal{L}_p f](x) = \frac{1}{p} \sum_{k=0}^{p-1} f\left(\frac{x+k}{p}\right)$$

while that for the Gauss map is

$$(3.6) \quad [\mathcal{L}_G f](x) = \sum_{k=1}^{\infty} \frac{1}{(x+k)^2} f\left(\frac{1}{1+x}\right)$$

This last is known as the Gauss-Kuzmin-Wirsing operator [give ref]. It is not studied further here, although it has bearing on some results, due to its relationship to the representation of the real numbers.

As indicated in the introduction a critical issue and a point of confusion is that the operator \mathcal{L}_B is not well-defined without also specifying the function space on which it acts, and without also specifying the topology to be used on the unit interval. In particular, the spectrum of eigenvalues and eigenvectors for \mathcal{L}_B can vary from being discrete, to being continuous, depending on the the function space and the topology. The simplest case assumes the natural topology of the reals on the unit interval, and takes as the function space the set of orthogonal polynomials on the unit interval. In this case, the eigenfunctions

may be shown to be the Bernoulli polynomials $B_n(x)$, associated with the eigenvalues 2^{-n} . That is, one finds that

$$(3.7) \quad [\mathcal{L}_B B_n](x) = \frac{1}{2^n} B_n(x)$$

where the first few $B_n(x)$ are

$$(3.8) \quad \begin{aligned} B_0(x) &= 1 \\ B_1(x) &= x - \frac{1}{2} \\ B_2(x) &= x^2 - x + \frac{1}{6} \\ B_3(x) &= x^3 - \frac{3x^2}{2} + \frac{x}{2} \end{aligned}$$

and so on. Perhaps the easiest proof that these are the eigenfunctions may be obtained by considering the generating function for the Bernoulli polynomials:

$$(3.9) \quad G(x, t) = \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

It is then straight-forward to verify that

$$(3.10) \quad [\mathcal{L}_B G](x, t) = \frac{1}{2} \left[G\left(\frac{x}{2}, t\right) + G\left(\frac{x+1}{2}, t\right) \right] = G\left(x, \frac{t}{2}\right)$$

or equivalently, by applying the linearity of \mathcal{L}_B , that

$$(3.11) \quad [\mathcal{L}_B G](x, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} [\mathcal{L}_B B_n](x) = \sum_{n=0}^{\infty} \left(\frac{t}{2}\right)^n \frac{B_n(x)}{n!}$$

and then equating the coefficients of the powers of t . This derivation follows for the general p -adic case:

Theorem 3.1. *The eigenvalues of the p -adic transfer operator are the Bernoulli polynomials, and are associated with the eigenvalues p^{-n} . That is, one has*

$$(3.12) \quad [\mathcal{L}_p B_n](x) = \frac{1}{p^n} B_n(x)$$

Proof. The proof proceeds as in the above 2-adic case, and hinges on the well-known factorization

$$w^p - 1 = (w - 1)(w^{p-1} + w^{p-2} + \dots + 1)$$

Here, take $w^p - 1 = e^t - 1$, so that

$$\begin{aligned} [\mathcal{L}_p G](x, t) &= \frac{1}{p} \sum_{k=0}^{p-1} G\left(\frac{x+k}{p}, t\right) \\ &= \frac{1}{p} \frac{t}{e^t - 1} \sum_{k=0}^{p-1} \exp\left(\frac{(x+k)t}{p}\right) \\ &= \frac{t}{p} \frac{\left(1 + e^{t/p} + e^{2t/p} + \dots + e^{(p-1)t/p}\right)}{e^t - 1} \\ &= G\left(x, \frac{t}{p}\right) \end{aligned}$$

Then, equating coefficients of powers of t of the generating function, one obtains the desired result. \square

The above theorem is a fancy restatement of an old and well-known result on the Bernoulli polynomials, namely, the so-called “multiplication theorem” given by Joseph Ludwig Raabe in 1851[need ref]. This is usually given in the more prosaic form of

$$(3.13) \quad B_n(px) = p^{n-1} \sum_{k=0}^{p-1} B_n\left(x + \frac{k}{p}\right)$$

but amounts to the same thing. What the language of the transfer operator provides is an abstraction that allows the multiplication theorem to be examined in a broader fashion. In particular, multiplication theorems exist not only for the Bernoulli polynomials, but more broadly, including the Gamma function and the Hurwitz zeta function. These will be re-discovered in later sections.

The following two sections provide an alternate and more abstract and labored derivation of the above result. The goal of the abstraction is to develop the machinery needed to explore the Bernoulli operator in more general topological settings. Rather than starting with the Hilbert space of orthogonal polynomials on the unit interval, the next section defines a Banach space on monomials, and its dual. The subsequent section then exposes the matrix elements of the Bernoulli operator in this space.

4. THE POLYNOMIAL REPRESENTATION

The polynomial eigenvectors of the Bernoulli operator can be derived in several ways. One seemingly natural approach is to start with a Hilbert space of orthogonal polynomials on the unit interval. This approach is taken by Driebe[?], who starts with the Legendre polynomials, rescaled to the unit interval, and obtains the resulting Bernoulli polynomials. This allows for a derivation of the right eigenvectors, but is then discovered to generate difficulties when considering the left eigenvectors. This arises because the Bernoulli operator is not invertible in this Hilbert space: it is quite singular, in that $[\mathcal{L}_B f] = 0$ whenever f is skew about $1/2$, that is, whenever $f(y) = -f(1/2 + y)$. A set of left eigenvectors can be found, but these are not polynomials or even ordinary real-valued functions; rather, they are generalized functions, expressed as derivatives of the Dirac delta function. From this exercise, one concludes that the starting assumption of envisioning the transfer operator acting on a Hilbert space does not provide any particular benefit or insight, and thus can be dispensed with.

Generalized functions can be loosely defined as linear functionals from the space of functions to the reals. In this sense, generalized functions belong to the dual space of a function space. Given that the left eigenvectors of the Bernoulli operator can be shown to belong to such a dual space, it then makes sense to start in this way. Thus, for the following, an infinite-dimensional vector space will be constructed, with the basis elements being the monomials. The generalized functions appear naturally as elements of the dual space. The transfer operator then has a direct representation in this space. In principle, this infinite-dimensional vector space may be taken to be a Banach space; however, the development below does not make any particular use of the norm that Banach spaces are equipped with.

This is perhaps a critical point, and deserves being belabored. In this section, the vocabulary for discussing an infinite-dimensional vector space will be developed. The sums appearing herein range formally over the countable infinity. However, the language of this section avoids questions of the convergence of these sums; the question of convergence can be deferred to later sections, when one is manipulating actual operators, and the sums take

a concrete form. Thus, for the development of this section, a discussion of the vector norm, a discussion of a metric on the vector space, and a discussion of the topology of the vector space can be completely avoided. The manipulations are completely algebraic in nature. Curiously, this will continue to be the case when these algebraic manipulations are applied to the Bernoulli operator. That is, the resulting concrete sums will turn out to range over only a finite number of non-zero terms, and so the question of convergence will not come up. As a result, the foundations do not require a notion of norm, metric or topology on this vector space. When the exceptions to this rule crop up, they will be discussed explicitly.

XXX the above is false, as there are plenty of places below where it is assumed that an infinite number of elts are non-zero. Again, need to refine the algebraic vs. topological discussion to handle these subtleties XXX.

Consider an infinite-dimensional vector space V with a countable ordered set of linearly independent basis vectors e_k labelled by the natural numbers k . A general element $v \in V$ may be written as $v = \sum_{k=0}^{\infty} a_k e_k$. The dual space V^* is the set of all linear functionals $L : V \rightarrow \mathbb{R}$. General elements of the dual space may be written as linear combinations of the basis elements e_k^* , which are the maps such that $e_j^*(e_k) = \delta_{jk}$.

For the space \mathcal{P} of real-analytic functions on the unit interval, the basis elements may be taken to be the monomials $e_k = x^k$, so that a general real-analytic function $f \in \mathcal{P}$ is written as

$$(4.1) \quad f(x) = \sum_{n=0}^{\infty} a_n x^n$$

This may be trivially interpreted as nothing more than the Taylor's series for a real-valued function expanded at $x = 0$. The correct formalism for discussing the dual space is a bit trickier. There is a strong historical desire to represent the linear operators of the dual space by means of integrals, that is, to represent the linear functional $L : \mathcal{P} \rightarrow \mathbb{R}$ by an integral

$$\int_0^1 l(x) f(x) dx$$

where $f \in \mathcal{P}$ and l is some "generalized function". Momentarily succumbing to this desire, one finds that the dual vectors may be written as

$$e_k^* = \frac{(-1)^k}{k!} \delta^{(k)}(x)$$

where $\delta(x)$ is the Dirac delta function. Thus, one has as the duality relation

$$(4.2) \quad e_j^*(e_k) = \int_0^1 x^k \frac{(-1)^j}{j!} \delta^{(j)}(x) dx = \int_0^1 \delta(x) \frac{d^j x^k}{dx^j} dx = \delta_{jk}$$

after integration by parts. Generalized functions are less than an ideal mechanism for representing elements of the dual space, but it is a mostly workable and consistent mechanism; its drawbacks in this particular context will be discussed in great detail below. The relation 4.2 above demonstrates orthogonality; one may also show completeness:

$$\sum_{n=0}^{\infty} e_n \otimes e_n^* = \sum_{n=0}^{\infty} x^n \frac{(-1)^n}{n!} \delta^{(n)}(x) = \delta(x-y)$$

The remainder of the text switches to the quantum mechanical bra-ket notation, writing $|n\rangle = e_n$ and $\langle n| = e_n^*$. The orthogonality condition becomes $\langle n| m\rangle = \delta_{nm}$ while for completeness one writes

$$\mathbb{I} = \sum_{n=0}^{\infty} e_n \otimes e_n^* = \sum_{n=0}^{\infty} |n\rangle \langle n|$$

The re-introduction of the coordinate x is done by writing $\langle x|m\rangle = x^m$, while for the transpose one writes $\langle n|x\rangle = (-1)^n \delta^{(n)}(x)/n!$. The advantage of the bra-ket notation over the use of the e_k is that it can be used to make clear when one is discussing the coordinate representation, involving x or $\delta(x)$, and when one is discussing the vector space elements in the abstract, without reference to the coordinate representation. To fully articulate this notation, one may write the Taylor's series as

$$\begin{aligned}
 f(x) &= \langle x|f\rangle \\
 &= \sum_{n=0}^{\infty} \langle x|n\rangle \langle n|f\rangle \\
 &= \sum_{n=0}^{\infty} x^n \langle n|f\rangle \\
 &= \sum_{n=0}^{\infty} x^n \int dy \langle n|y\rangle \langle y|f\rangle \\
 &= \sum_{n=0}^{\infty} x^n \int dy (-1)^n \frac{\delta^{(n)}(y)}{n!} f(y) \\
 (4.3) \quad &= \sum_{n=0}^{\infty} x^n \frac{f^{(n)}(0)}{n!}
 \end{aligned}$$

This notation allows the matrix elements $U_{mn} = \langle m|\mathcal{L}|n\rangle$ of the transfer operator to be articulated as well. Starting with the classical notation of equation 4.1, one has

$$[\mathcal{L}f](x) = \sum_{m=0}^{\infty} b_m x^m = \sum_{m=0}^{\infty} x^m \sum_{n=0}^{\infty} U_{mn} a_n$$

or, equivalently

$$[\mathcal{L}f](x) = \langle x|\mathcal{L}f\rangle = \sum_{m=0}^{\infty} \langle x|m\rangle \langle m|\mathcal{L}f\rangle = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle x|m\rangle \langle m|\mathcal{L}|n\rangle \langle n|f\rangle$$

where $\langle n|f\rangle = a_n$ and $\langle n|\mathcal{L}f\rangle = b_n$. Equating each power of x^m one finds, in the classical notation, that

$$\frac{1}{m!} \left. \frac{d^m [\mathcal{L}f](x)}{dx^m} \right|_{x=0} = \sum_{n=0}^{\infty} U_{mn} \frac{1}{n!} \left. \frac{d^n f(x)}{dx^n} \right|_{x=0}$$

where

$$\langle n|f\rangle = a_n = \frac{1}{n!} \left. \frac{d^n f(x)}{dx^n} \right|_{x=0}$$

and so on. The formulation of the transfer operator given in equation 2.1 fits in this framework as well; it is merely the spatial representation of the matrix elements:

$$\begin{aligned}
 \delta(x-g(y)) &= \mathcal{L}(x,y) \\
 &= \langle x|\mathcal{L}|y\rangle = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle x|m\rangle \langle m|\mathcal{L}|n\rangle \langle n|y\rangle
 \end{aligned}$$

Armed with this notation, the following section explores the matrix elements of the Bernoulli operator in this polynomial basis.

5. THE BERNOULLI OPERATOR IN THE POLYNOMIAL BASIS

This section reviews the structure of the Bernoulli operator \mathcal{L}_B in the polynomial basis developed above. The matrix elements in the monomial basis are given by

$$(5.1) \quad [\mathcal{L}_B]_{mk} \equiv U_{mk} \equiv \langle m|U|k\rangle = \frac{\delta_{mk}}{2^m} + \binom{k}{m} \frac{\Theta_{mk}}{2^{k+1}}$$

where $\binom{m}{k} = \frac{m!}{k!(m-k)!}$ is the binomial coefficient, and

$$\Theta_{mk} = \begin{cases} 0 & \text{if } k \leq m \\ 1 & \text{if } k > m \end{cases}$$

is a traceless, pure upper-triangular matrix. These matrix elements are easily obtained by direct substitution, that is, by contemplating the coefficient to the x^m term in the expansion of

$$[\mathcal{L}_B x^k] = \frac{1}{2} \left[\left(\frac{x}{2}\right)^k + \left(\frac{1+x}{2}\right)^k \right]$$

Both the diagonal elements, and the reason for the appearance of the binomial coefficients should be immediately clear. Note that the evaluation of these matrix elements does not require the evaluation of any sums with an infinite number of non-zero terms.

The eigenvalues may be promptly read off the diagonal; these eigenvalues are $\lambda_n = 2^{-n}$. Because the matrix is upper-triangular, it is easily solvable for both the left and right eigenvectors, which agree perfectly with those given by Driebe[?]. Visually, the upper-left of this matrix looks like

$$U_{mk} = \begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \dots \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \dots \\ 0 & 0 & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \dots \\ 0 & 0 & 0 & \frac{1}{8} & \frac{1}{16} & \dots \\ 0 & 0 & 0 & 0 & \frac{1}{16} & \dots \\ \dots & & & & & \dots \end{bmatrix}$$

The right and left eigenvectors are developed in the following sections.

5.1. Right eigenvectors. The right eigenvectors are denoted by $|B_n\rangle$ and have the vector components

$$(5.2) \quad \langle k|B_n\rangle = \binom{n}{k} (1 - \Theta_{n,k}) B_{n-k} = \begin{cases} \binom{n}{k} B_{n-k} & \text{for } k \leq n \\ 0 & \text{for } k > n \end{cases}$$

where B_k are the Bernoulli numbers. Note that only a finite number of these vector components are non-vanishing.

Theorem 5.1. *The $|B_n\rangle$ are eigenvectors of U_{mk} , associated with eigenvalues 2^{-n} . That is,*

$$\mathcal{L}_B |B_n\rangle = \frac{1}{2^n} |B_n\rangle$$

Proof. This may be verified in the monomial basis through brute-force multiplication of the vector into the matrix:

$$\begin{aligned} \sum_{k=0}^{\infty} \langle m|U|k\rangle \langle k|B_n\rangle &= \sum_{k=m}^n \left[\frac{\delta_{mk}}{2^m} + \binom{k}{m} \frac{\Theta_{mk}}{2^{k+1}} \right] \binom{n}{k} B_{n-k} \\ &= \frac{1}{2^m} \binom{n}{m} B_{n-m} + \dots \text{non-trivial-taylor-expn} \\ &= \frac{1}{2^n} \langle m|B_n\rangle \end{aligned}$$

XXX fill in details. Note that this proof does not require the evaluation of any sums with an infinite number of non-zero elements; all sums are algebraically finite. \square

The right eigenvectors can be given a representation in coordinate space, and these are found to be the Bernoulli polynomials discussed previously:

$$\sum_{k=0}^{\infty} \langle x|k\rangle \langle k|B_n\rangle = \sum_{k=0}^n \binom{n}{k} B_{n-k} x^k = B_n(x)$$

These were previously shown to be eigenvectors, and so this is consistent with the above. Thus, at this level, one may conclude that the coordinate representation and the matrix representation for this transfer operator are consistent.

5.2. Left Eigenvectors. This matrix expression for U_{mn} also admits left eigenvectors which can be given an explicit representation. Letting the left eigenvectors be denoted by $\langle \tilde{B}_n|$, they have, for $n > 0$, the components

$$(5.3) \quad \langle \tilde{B}_n|k\rangle = \binom{k}{n-1} \frac{1}{n} (\delta_{nk} + \Theta_{nk})$$

The zeroth left eigenvector is a special-case; it has components $\langle \tilde{B}_0|k\rangle = 1/(k+1)$. Unlike the right eigenvectors, the left eigenvectors all have an infinite number of non-zero components. They share the same eigenvalue spectrum with the right eigenvectors, so that

$$\sum_{k=0}^{\infty} \langle \tilde{B}_n|m\rangle U_{mk} = \frac{1}{2^n} \langle \tilde{B}_n|k\rangle$$

which may again be demonstrated by a brute-force evaluation of the sum. Despite the fact that an infinite number of the left eigenvector components are non-vanishing, this sum will only contain a finite number of non-zero terms, and thus its finiteness is guaranteed on algebraic grounds.

One may write down a coordinate-space representation for the left eigenvectors, by contracting them against the dual-space basis elements $\langle k|x\rangle$. This leads directly to the generalized functions:

$$\begin{aligned} \langle \tilde{B}_n|x\rangle &= \sum_{k=0}^{\infty} \langle \tilde{B}_n|k\rangle \langle k|x\rangle \\ &= \sum_{k=0}^{\infty} \langle \tilde{B}_n|k\rangle (-)^k \frac{\delta^{(k)}(x)}{k!} \\ &= \frac{1}{n} \sum_{k=n}^{\infty} \binom{k}{n-1} (-)^k \frac{\delta^{(k)}(x)}{k!} \\ (5.4) \quad &= \frac{(-)^{n+1}}{n!} \left[\delta^{(n-1)}(1-x) - \delta^{(n-1)}(x) \right] \end{aligned}$$

for the $n > 0$ case. The $n = 0$ left eigenvector is best understood by integrating it over some arbitrary function $f(x)$:

$$\begin{aligned}
 \int_0^1 dx \langle \tilde{B}_0 | x \rangle f(x) &= \int_0^1 dx f(x) \sum_{k=0}^{\infty} (-)^k \frac{\delta^{(k)}(x)}{(k+1)!} \\
 &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{(k+1)!} \\
 &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \int_0^1 x^k dx \\
 &= \int_0^1 f(x) dx \\
 (5.5) \qquad \qquad \qquad &= \langle \tilde{B}_0 | f \rangle
 \end{aligned}$$

The other left eigenvectors can also be made more concrete by looking at how they act on some function $f(x)$; this may be written as

$$(5.6) \qquad \qquad \qquad \langle \tilde{B}_n | f \rangle = \frac{1}{n!} \left[f^{(n-1)}(1) - f^{(n-1)}(0) \right]$$

5.3. Duality. That the left eigenvectors are dual to the Bernoulli polynomials, which can be verified in either the matrix-element basis, or the coordinate-space representation.

Theorem 5.2. *The left and right eigenvectors are dual, in that*

$$\langle \tilde{B}_n | B_m \rangle = \sum_{k=0}^{\infty} \langle \tilde{B}_n | k \rangle \langle k | B_m \rangle = \delta_{nm}$$

and furthermore, the duality is algebraic, in that no infinite sums need be performed to demonstrate duality.

Proof. Consider first the $n = 0$ case. One has

$$\begin{aligned}
 \langle \tilde{B}_0 | B_m \rangle &= \sum_{k=0}^{\infty} \langle \tilde{B}_0 | k \rangle \langle k | B_m \rangle \\
 &= \sum_{k=0}^m \frac{1}{k+1} \binom{m}{k} B_{m-k} \\
 &= \frac{1}{m+1} \sum_{j=0}^m \binom{m+1}{j} B_j \\
 &= \delta_{m0}
 \end{aligned}$$

where the substitution $j = m - k$ was made to obtain the last sum. The vanishing of the last sum is a well-known identity on the Bernoulli numbers. The $n \neq 0$ case requires more

work, but ends similarly:

$$\begin{aligned}
\langle \tilde{B}_n | B_m \rangle &= \sum_{k=0}^{\infty} \langle \tilde{B}_n | k \rangle \langle k | B_m \rangle \\
&= \sum_{k=0}^m \binom{k}{n-1} \frac{1}{n} (\delta_{nk} + \Theta_{nk}) \binom{m}{k} B_{m-k} \\
&= \frac{1}{n} \binom{n}{n-1} \binom{m}{n} B_{m-n} + \sum_{k=n+1}^m \binom{k}{n-1} \frac{1}{n} \binom{m}{k} B_{m-k} \\
&= \begin{cases} 0 & \text{for } n > m \\ B_0 & \text{for } n = m \\ \binom{m}{n} B_{m-n} + \frac{m!}{n!(m-n+1)!} \sum_{k=n+1}^m \binom{m-n+1}{m-k} B_{m-k} & \text{for } n < m \end{cases}
\end{aligned}$$

The last case can be shown to vanish, by again making the substitution $j = m - k$, to get

$$\begin{aligned}
\binom{m}{n} B_{m-n} + \frac{m!}{n!(m-n+1)!} \sum_{j=0}^{m-n-1} \binom{m-n+1}{j} B_j \\
= \frac{m!}{n!(m-n+1)!} \sum_{j=0}^{m-n} \binom{m-n+1}{j} B_j = 0
\end{aligned}$$

thus concluding the proof. Notice that this proof does not require the evaluation of any infinite sums: all sums are performed over a finite number of terms. \square

This proof of duality may also be conducted in coordinate space, where it takes the form

$$\begin{aligned}
\langle \tilde{B}_n | B_m \rangle &= \int_0^1 \langle \tilde{B}_n | x \rangle \langle x | B_m \rangle dx \\
&= \int_0^1 \frac{(-1)^{n+1}}{n!} [\delta^{(n-1)}(1-x) - \delta^{(n-1)}(x)] B_m(x) dx \\
&= \begin{cases} 0 & \text{for } m < n-1 \\ \frac{m!}{n!(m-n+1)!} [B_{m-n+1}(1) - B_{m-n+1}(0)] & \text{for } m \geq n-1 \end{cases} \\
(5.7) \quad &= \delta_{mn}
\end{aligned}$$

and so, as with most of the previous results, one may be lulled into a sense of complacency about the equivalence of the coordinate-space and monomial-vector-space representations. This complacency is ill-founded, as demonstrated below.

5.4. Completeness. Given this duality, the operator

$$(5.8) \quad \mathbb{I}_B = \sum_{n=0}^{\infty} |B_n\rangle \langle \tilde{B}_n|$$

can then be recognized as a projection operator. In fact, it is complete in the monomial basis, in that

$$(5.9) \quad \langle j | \mathbb{I}_B | k \rangle = \langle j | \sum_{n=0}^{\infty} |B_n\rangle \langle \tilde{B}_n | k \rangle = \delta_{jk}$$

can be shown, using essentially the same operations as in the proof above. Also, as before, this demonstration involves sums with only a finite number of terms, and so completeness

may be taken as an algebraic property. That is, \mathbb{I}_B may be taken to be the identity operator on the vector space of polynomials.

Curiously, this identity operator \mathbb{I}_B expanded in the Bernoulli basis as in formula 5.8 is the Euler-Maclaurin summation formula in disguise. This may be seen by expanding

$$\begin{aligned}
 f(x) &= \langle x|f \rangle \\
 (5.10) \quad &= \sum_{m=0}^{\infty} B_m(x) \langle \tilde{B}_m|f \rangle \\
 &= \int_0^1 f(y) dy + \sum_{m=0}^{\infty} \frac{B_m(x)}{m!} [f^{(m-1)}(1) - f^{(m-1)}(0)]
 \end{aligned}$$

This may be compared to the $n = 1$ case of the traditional Euler-Maclaurin summation formula,

$$\frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k+x}{n}\right) = \int_0^1 f(y) dy + \sum_{m=0}^{\infty} \frac{B_m(x)}{n^m m!} [f^{(m-1)}(1) - f^{(m-1)}(0)]$$

By combining equations 5.85.9 and 5.10 one is sorely tempted to deduce orthogonality over coordinate space. That is, one wants to deduce that

$$(5.11) \quad \sum_{n=0}^{\infty} \langle x|B_n \rangle \langle \tilde{B}_n|y \rangle = \delta(x-y)$$

which superficially seems to be entirely reasonable. The problem with equation 5.11 is that it is misleading, as will be expanded upon in the following sections. As long as the functions $f(x)$ are in all cases understood to be polynomials, then there is no harm in using equation 5.11. This follows in part because \mathbb{I}_B really is the identity operator on the space of polynomials: there are no polynomials (other than $f(x) = 0$) that are in the kernel of \mathbb{I}_B . However, one may consider larger function spaces than those consisting of polynomials; on these spaces, \mathbb{I}_B will be found to have a large and non-trivial kernel, while by contrast, the Dirac delta function $\delta(x-y)$ does not have a non-trivial kernel on these same spaces. This will be delved into in the next section.

5.5. The Bernoulli operator in diagonal form; the Koopman operator. From the above manipulations, one may deduce that, in the polynomial representation, the Frobenius-Perron operator of the Bernoulli map is

$$\mathcal{L}_B = \sum_{n=0}^{\infty} |B_n \rangle \lambda_n \langle \tilde{B}_n|$$

We can make use of this diagonal form to easily compute formal expressions involving \mathcal{L}_B . Thus, for a function $f(x)$ that is expressible as a polynomial series in x , one may write the operator

$$f(\mathcal{L}_B) = \sum_{n=0}^{\infty} |B_n \rangle f(\lambda_n) \langle \tilde{B}_n|$$

whose matrix elements can be explicitly demonstrated in the monomial basis:

$$\langle j|f(\mathcal{L}_B)|k \rangle = \sum_{j \leq n \leq k} \binom{n}{j} \binom{k}{n-1} \frac{B_{n-j}}{n} f(2^{-n})$$

As in previous cases, note that the summation involves only a finite number of terms, and is thus manifestly finite (provided that f is finite). As curious example, one may write, $\mathcal{L}_B = \exp(-H_B)$ so that $H_B = -\log \mathcal{L}_B$ has matrix elements

$$\langle j | H_B | k \rangle = \frac{\log(2)}{k+1} \binom{k+1}{j} \sum_{m=0}^{k-j} \binom{k-j+1}{m} (j+m) B_m$$

None of the eigenvalues λ_n are zero. In the previous section, it was shown that the kernel of \mathbb{I}_B is trivial. Thus, \mathcal{L}_B is invertible. This inverse is known as the *Koopman operator*, and is denoted by \mathcal{K}_B :

$$\mathcal{K}_B = \sum_{n=0}^{\infty} |B_n\rangle \frac{1}{\lambda_n} \langle \tilde{B}_n|$$

By duality and completeness, one has that the Koopman operator is both a left and a right inverse,

$$\mathcal{L}_B \mathcal{K}_B = \mathcal{K}_B \mathcal{L}_B = \mathbb{I}_B$$

in the polynomial representation. In this representation, one may honestly write $\mathcal{K}_B = \mathcal{L}_B^{-1}$. This will not at all be the case when one considers the Bernoulli operator \mathcal{L}_B acting on other function spaces: it will be seen to have a large and non-trivial kernel, and so it will not be invertible (as “half” of its eigenvalues will be seen to be zero).

5.6. Change of Basis Recap. This section simply recaps the previous results. It was seen above that the monomials form a complete set of basis states that can be used to represent polynomials. The operator $\mathbb{I}_M = \sum_{n=0}^{\infty} |n\rangle \langle n|$ can be called the identity operator over the space of polynomials; it has no non-trivial kernel in that space. Here, the subscript M is used to indicate that the identity operator is built from the monomial states. Thus, for the Bernoulli operator \mathcal{L}_B , one may confidently write $\mathcal{L}_B = \mathbb{I}_M \mathcal{L}_B \mathbb{I}_M$ which expands to

$$\mathcal{L}_B = \mathcal{L}_B^{\text{Monomial}} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |m\rangle \langle m | \mathcal{L}_B | n\rangle \langle n|$$

The superscript “Monomial” is a formal label, used only to emphasize the basis in which the operator was expanded in. As seen above, the matrix elements $U_{mn} \equiv \langle m | \mathcal{L}_B | n\rangle$ are upper-triangular.

The operator $\mathbb{I}_B = \sum_{n=0}^{\infty} |B_n\rangle \langle \tilde{B}_n|$ is also the identity operator on the space of polynomials. Using the same trick to write

$$\mathcal{L}_B = \mathcal{L}_B^{\text{Bernoulli}} = \mathbb{I}_B \mathcal{L}_B \mathbb{I}_B = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |B_m\rangle \langle \tilde{B}_m | \mathcal{L}_B | B_n\rangle \langle \tilde{B}_n|$$

one finds that $\mathcal{L}_B^{\text{Bernoulli}}$ is an operator which has matrix elements that are non-zero only on the diagonal: $\langle B_m | \mathcal{L}_B | B_n\rangle = \delta_{mn} \lambda_n$.

Consider now the change of basis from the monomial basis to the Bernoulli basis. Explicitly, this change of basis is

$$\begin{aligned} \mathcal{L}_B^{\text{Bernoulli}} &= \sum_{j,k=0}^{\infty} |B_j\rangle \delta_{jk} \lambda_k \langle \tilde{B}_k| \\ &= \sum_{j,k,m,n=0}^{\infty} |B_j\rangle \langle \tilde{B}_j | m\rangle \langle m | \mathcal{L}_B^{\text{Monomial}} | n\rangle \langle n | B_k\rangle \langle \tilde{B}_k| \\ (5.12) \quad &= \tilde{B} \mathcal{L}_B^{\text{Monomial}} B \end{aligned}$$

where the operators \tilde{B} and B show the change of basis:

$$\tilde{B} = \sum_{j,m=0}^{\infty} |B_j\rangle \langle \tilde{B}_j| m\rangle \langle m|$$

and

$$B = \sum_{k,n=0}^{\infty} |n\rangle \langle n| B_k\rangle \langle \tilde{B}_k|$$

It is not hard to work out that $\tilde{B}B = \mathbb{I}_B$ and that $B\tilde{B} = \mathbb{I}_M$ so \tilde{B} is both a left- and right-inverse of B ; one may confidently write $B^{-1} = \tilde{B}$ as a two-sided inverse. Although B and \tilde{B} are inverses of one-another, they are in no way orthogonal. The matrix elements of the transpose of an orthogonal operator are equal to those of the inverse; a quick review of equations 5.2 and 5.3 makes it clear that $\langle \tilde{B}_k| n\rangle \neq \langle n| B_k\rangle$, and so B is not an orthogonal operator. This is not a surprise: orthogonal operators cannot take a diagonal operator and make it triangular.

5.7. Operator recap. XXX Talk about the operator

$$\sum_n \lambda_n e_n \otimes e_n^*$$

in terms of being a compact operator, a nuclear operator, etc. Talk about the order of the operator, Talk about the projective topological tensor product, in the language of Grothendieck.

6. TOPOLOGY, COMPLETENESS AND ORTHOGONALITY

The notions of completeness and orthogonality are treated above without any appeal to topology. That is, they are handled with what is essentially an algebraic approach, where all sums are essentially finite and well defined because all sums involve only a finite number of non-zero terms. This was possible in part by construction, and in part by luck: the Bernoulli operator was a solvable, upper-triangular matrix in the infinite vector space whose basis elements are the monomials. A sum over monomials, where only a finite number of terms are non-zero, is a polynomial. To go beyond this, to get to more general functions, such as, for example, a sum over monomials with an infinite number of non-zero elements, requires the introduction of a topology on the infinite vector space, so that limits of Cauchy sequences can be defined and discussed. There are several ways to provide a topology; the straightforward way is to provide the space with a metric topology. A metric topology endows the infinite vector space with a norm, so that the length of a vector can be given, and the distance between vectors defined as the length of the vector difference.

With this in mind, the question then turns to “what are the interesting topologies?”. Before this question is asked in earnest, it is worth illustrating why, exactly, it is an important question, and why the role of topologies needs to be addressed. Some of the difficulties of sticking to a purely algebraic approach are illustrated in this section.

The operator $\mathbb{I}_B = \sum_{n=0}^{\infty} |B_n\rangle \langle \tilde{B}_n|$ was shown to be complete on the space of polynomials. By this, it is meant that there is no polynomial that lies in the kernel of \mathbb{I}_B , other than the trivial polynomial $p(x) = 0$. Equivalently, there does not exist any polynomial for which the identity $p(x) = -p(x+1/2)$ holds: more broadly, there is no such thing as a periodic polynomial. A function which obeys $f(x) = -f(x+1/2)$ is of necessity periodic, of which $f(x) = \sin(2\pi x)$ is a canonical example. More generally, sine and cosine waves which have an odd number of periods in the unit interval are all in the kernel of the

Bernoulli operator, when that operator is taken in the coordinate-space. This may be seen very easily simply by direct substitution into equation 3.4, which promptly yields

$$(6.1) \quad \mathcal{L}_B^{\text{Coordinate}} \sin 2\pi(2k+1)x = \frac{1}{2} [\sin \pi(2k+1)x + \sin \pi(2k+1)(x+1)] = 0$$

for integers k ; likewise for the cosine. Not so for waves with an even number of periods:

$$\mathcal{L}_B^{\text{Coordinate}} \sin 4\pi kx = \frac{1}{2} [\sin 2\pi kx + \sin 2\pi k(x+1)] = \sin 2\pi kx$$

Here, the superscript ‘‘Coordinate’’ is introduced to distinguish the operator in the coordinate basis, as given in equation 3.4, from the same operator in the monomial basis.

In the previous sections, it was established (XXX but perhaps not very clearly? XXX) that $\mathcal{L}_B^{\text{Bernoulli}} = \mathcal{L}_B^{\text{Monomial}}$ when considered as operators acting on the space of polynomials. However, these are not at all equivalent if one considers them as operators acting on sines and cosines. In particular, consider \mathbb{I}_B acting on $f(x) = \exp 2\pi ikx$:

$$(6.2) \quad \begin{aligned} [\mathbb{I}_B f](x) &= \langle x | \mathbb{I}_B | f \rangle \\ &= \sum_{n=0}^{\infty} \langle x | B_n \rangle \langle \tilde{B}_n | f \rangle \\ &= \sum_{n=0}^{\infty} \langle x | B_n \rangle \frac{1}{n!} [f^{(n-1)}(1) - f^{(n-1)}(0)] \\ &= \sum_{n=0}^{\infty} \langle x | B_n \rangle \frac{1}{n!} [(2\pi ik)^{n-1} e^{2\pi ik} - (2\pi ik)^{n-1}] \\ (6.3) \quad &= 0 \end{aligned}$$

where the second step makes use of equation 5.6. This is remarkable, as any periodic wave constructed from sines and cosines seems to be in the kernel. By contrast, \mathbb{I}_M does not behave this way:

$$(6.4) \quad \begin{aligned} [\mathbb{I}_M f](x) &= \langle x | \mathbb{I}_M | f \rangle \\ &= \sum_{n=0}^{\infty} \langle x | n \rangle \langle n | f \rangle \\ &= \sum_{n=0}^{\infty} \langle x | n \rangle \frac{1}{n!} f^{(n)}(0) \\ &= \sum_{n=0}^{\infty} \langle x | n \rangle \frac{1}{n!} (2\pi ik)^n \\ (6.5) \quad &= \sum_{n=0}^{\infty} x^n \frac{1}{n!} (2\pi ik)^n \\ (6.6) \quad &= \exp 2\pi ikx \end{aligned}$$

which is just a re-derivation of equation 4.3. Thus, \mathbb{I}_B and \mathbb{I}_M are inequivalent when acting on sine functions; so $\mathcal{L}_B^{\text{Monomial}}$ and $\mathcal{L}_B^{\text{Bernoulli}}$ are inequivalent as well.

Since the sine function is the limit of a polynomial sequence, it seems strange or somehow contradictory that $\mathcal{L}_B^{\text{Bernoulli}}$ has only a trivial kernel on the space of polynomials, while utterly and completely killing all sine functions. In order to define limits, or more precisely, in order to define $\sin 2\pi x$ as the limit of a sequence of polynomials, one must define the manner in which a polynomial sequence can converge to a function, and, for that, one must have a topology.

Can this conundrum be escaped without appealing to topology? Since \mathbb{I}_B seems to be somehow incomplete when considering sine functions, perhaps, one might think, this lack of completeness is due to the form of the left eigenstates given in equation 5.4. One might make a guess that perhaps a more truly complete set of states can be found by considering

$$\tilde{S}_n(x) = \frac{(-)^{n+1}}{n!} \left[\delta^{(n-1)}(1-x) + \delta^{(n-1)}(x) \right]$$

so that sums and differences of the duals $\tilde{B}_m(x)$ and $\tilde{S}_n(x)$ can be used to regain the duals to the monomials $\langle n|y\rangle = (-)^n \delta^{(n)}(y)/n!$.

Theorem 6.1. *The duals to $\tilde{S}_n(x)$ are given by $S_n(x) = nE_n(x)/2$ where the $E_n(x)$ are the Euler polynomials.*

Proof. Consider the generating function for the Euler polynomials

$$G_E(x,t) = \frac{2e^{xt}}{1+e^t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}$$

Then one has, by taking the left hand side,

$$\int_0^1 \tilde{S}_n(x) \frac{2e^{xt}}{1+e^t} dx = \frac{2}{n} \frac{t^{n-1}}{(n-1)!}$$

and, performing the same operation on the right hand side,

$$\int_0^1 \tilde{S}_n(x) \sum_{k=0}^{\infty} E_k(x) \frac{t^k}{k!} dx = \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_0^1 \tilde{S}_n(x) E_k(x) dx$$

Then, equating the two sides, one has demonstrated duality of these states:

$$\int_0^1 \tilde{S}_n(x) E_k(x) dx = \frac{2\delta_{k,n-1}}{n}$$

which completes the proof. \square

As with the Bernoulli polynomials, one can, to a limit extent, make a restricted completeness statement in coordinate space. That is, if one decomposes a function $f(y) = \sum_{k=1}^{\infty} a_k S_k(y)$ then one easily finds

$$\int_0^1 \sum_{n=1}^{\infty} S_n(x) \tilde{S}_n(y) f(y) dy = f(x)$$

from which one wants to conclude, once again incorrectly, or at least misleadingly, that

$$\sum_{n=1}^{\infty} S_n(x) \tilde{S}_n(y) = \delta(x-y)$$

by repeating the same concerns and issues that lead to equation 5.11. The fault is the assumption that arbitrary, non-polynomial $f(y)$ can be decomposed in the fashion given. In fact, the operator $\mathbb{I}_S = \sum_{n=0}^{\infty} |S_n\rangle \langle \tilde{S}_n|$ has a large kernel: this time, all functions that are evenly periodic are in the kernel. That is, any function for which one has $f(y) = f(y+1/2)$ lies in the kernel of \mathbb{I}_S .

One might hope that one can remedy the above situation by taking the sum $\mathbb{I}_C = \mathbb{I}_B + \mathbb{I}_S$ with one operator projecting out the even periodic functions, and the other the odd periodic functions, and that somehow, between the two of them, making a whole. However, one immediately runs into a problem with the basis functions. The $\tilde{S}_n(x)$ are not orthogonal to

the $B_n(x)$, and vice-versa. This is easily seen by considering the the generating function for the Bernoulli polynomials:

$$G_B(x,t) = \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

Integrating, one gets

$$\int_0^1 \tilde{S}_n(x) G_B(x,t) dx = \frac{t^n e^t + 1}{n! e^t - 1} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_0^1 \tilde{S}_n(x) B_k(x) dx$$

from which one deduces that $\int_0^1 \tilde{S}_n(x) B_k(x) dx = 0$ only for $k < n - 1$, and similarly for $\int_0^1 \tilde{B}_n(x) S_k(x) dx$.

More generally, any such attempt to patch up the situation is doomed to fail. This is a well-known theorem of functional analysis, which states that the space \mathbb{R}^ω does not have a countable algebraic basis. That is, while one may be able to write down a countable number of linearly independent vectors e_k with $k \in \mathbb{N}_0$, the linear combinations $\sum_k a_k e_k$ with only a finite number of $a_k \in \mathbb{R}$ being non-zero fail to span all of \mathbb{R}^ω . In particular, the vector $v = \sum_k e_k$ cannot be expressed as the sum over only a finite number of e_k . XXX Does this theorem have a name? Need to reference a ref for this. XXXX

7. THE FOURIER REPRESENTATION

The Koopman operator of the Bernoulli Map has the property of taking a function and making two copies of it. That is,

$$\begin{aligned} [\mathcal{K}_B f](y) &= \int_0^1 \delta(x - b(y)) f(x) dx \\ &= f(b(y)) \\ (7.1) \quad &= f(2y)\theta(1 - 2y) + f(2y - 1)\theta(2y - 1) \end{aligned}$$

where $\theta(x)$ is the step function, identically zero for $x < 0$ and identically one for $x > 0$. The Koopman operator for the Bernoulli map is not faithfully representable in the polynomial basis; this can be seen in two ways. First, it introduces a discontinuity at $x = 1/2$ which the polynomials cannot move beyond; the radius of the circle of converge is limited by this singularity. Secondly, it takes a function and more-or-less makes it periodic; again, the polynomials cannot cope directly with this. Thus, one is motivated to explore the Fourier representation, if only to express the Koopman operator.

It is easy to find an explicit form for this operator in the Fourier basis. Writing

$$f(x) = \sum_n a_n \cos 2\pi n x + b_n \sin 2\pi n x$$

then

$$[\mathcal{K}_B f](x) = \sum_n a_n \cos 4\pi n x + b_n \sin 4\pi n x$$

or, in Dirac notation, $\langle em | \mathcal{K}_B | en \rangle = \delta_{2m,n}$. This is a very singular operator in this basis. Visually, it has the distinctive appearance of

$$\mathcal{K}_B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & \\ 0 & 1 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 1 & 0 & \dots & \\ \dots & & & & & \end{bmatrix}$$

where every other row consists of zeros. The row-column numbering is such that the upper-left hand cell is numbered $(0,0)$, so that the next nonzero cell $(2,1)$ is the third from the top, and second over. In this same basis, \mathcal{L}_B is equally remarkable: it is literally the transpose: that is $\mathcal{K}_B = \mathcal{L}_B^T$ in this basis, and so

$$(7.2) \quad \mathcal{L}_B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \dots & \\ \dots & & & & & \end{bmatrix}$$

In this representation, it is easily seen that $\mathcal{L}_B \mathcal{K}_B = 1$ but $\mathcal{K}_B \mathcal{L}_B \neq 1$, just as in the coordinate-space representation. This can be used to show the following:

Theorem 7.1. *There does not exist a left inverse for \mathcal{L}_B .*

Proof. Assume that there exists an operator \mathcal{M} such that $\mathcal{M} \mathcal{L}_B = 1$. Then, by associativity of operator multiplication, one has

$$\mathcal{M} = \mathcal{M} \cdot 1 = \mathcal{M} \cdot (\mathcal{L}_B \mathcal{K}_B) = (\mathcal{M} \mathcal{L}_B) \cdot \mathcal{K}_B = 1 \cdot \mathcal{K}_B = \mathcal{K}_B$$

which is clearly false. Thus, such an \mathcal{M} does not exist. \square

It is instructive to verify that the Bernoulli polynomials are still eigenfunctions in this representation. For $n \neq 0$, one has

$$\int_0^1 B_1(x) \sin(2\pi n x) dx = \frac{-1}{\pi n}$$

and it is straightforward to visually verify that $U_B B_1 = \frac{1}{2} B_1$. By working with the generator for the Bernoulli polynomials,

$$\frac{t e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

one can immediately find, for $m \neq 0$, the Fourier components

$$\int_0^1 B_n(x) \cos(2\pi m x) dx = \begin{cases} 0 & \text{for } n \text{ odd} \\ (-1)^{1+n/2} n! / (2\pi m)^n & \text{for } n \text{ even} \end{cases}$$

and

$$\int_0^1 B_n(x) \sin(2\pi m x) dx = \begin{cases} 0 & \text{for } n \text{ even} \\ (-1)^{(n+1)/2} n! / (2\pi m)^n & \text{for } n \text{ odd} \end{cases}$$

Applying the Fourier-representation \mathcal{L}_B to these to these vector components makes it immediately clear how the eigenvalue of $1/2^n$ is associated with the eigenvector B_n .

7.1. The Hurwitz zeta eigenfunctions. The Fourier representation also makes it clear that any vector with vector components $a_n = 1/n^s$ will be an eigenvector of \mathcal{L}_B associated with the eigenvalue $\lambda = 1/2^s$. In coordinate space, one may write these eigenfunctions as

$$\beta(x; s) = 2\Gamma(s+1) \sum_{n=1}^{\infty} \frac{\exp(2\pi i n x)}{(2\pi n)^s}$$

which transform as $\mathcal{L}_B \beta(x; s) = 2^{-s} \beta(x; s)$. Up to the overall normalization, this function is sometimes called the “periodic zeta function”[?, Sect. 12.7]. Although it superficially

appears to be periodic in x , it is not actually so[?]: the value of $x = 1$ corresponds the branch point of the polylogarithm. The classical polylogarithm is defined as

$$\text{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$$

and so

$$\beta(x; s) = \frac{2\Gamma(s+1)}{(2\pi)^s} \text{Li}_s(e^{2\pi ix})$$

Although the polylogarithm can be analytically continued to all values of s and z , it has branch points at $z = 0$ and $z = 1$, and a non-trivial monodromy group[?].

This series recreates the Bernoulli polynomials for integer values of n , so for example, $\Re\beta(x; 2) = B_2(x)$ and $\Im\beta(x; 3) = B_3(x)$ and generally $\Re[(-i)^n \beta(x; n)] = -B_n(x)$. Equivalently, the Fourier series for the Bernoulli polynomials can be written as

$$\begin{aligned} (7.3) \quad B_n(x) &= -\Gamma(n+1) \sum_{k=1}^{\infty} \frac{\exp(2\pi ikx) + (-1)^n \exp(2\pi ik(1-x))}{(2\pi ik)^n} \\ &= \frac{-(-i)^n}{2} (\beta(x; n) + (-1)^n \beta(1-x; n)) \end{aligned}$$

See, for example [?, Thm. 12.19].

The periodic zeta function $\beta(x; s)$ is also an eigenfunction of the general p -adic operator, given in equation 3.5. That is, one has

$$(7.4) \quad \mathcal{L}_p \beta(x; s) = \frac{1}{p^s} \beta(x; s)$$

which may be demonstrated easily enough.

These eigenfunctions are essentially a form of the Hurwitz zeta function

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}$$

and that, in fact, the Hurwitz zeta itself is an eigenfunction, with eigenvalue 2^{s-1} . The Hurwitz zeta is related to β as

$$\beta(x; s) = \frac{is}{\sin \pi s} \left[e^{-i\pi s/2} \zeta(1-s, x) - e^{i\pi s/2} \zeta(1-s, 1-x) \right]$$

This relationship is known as Jonquière's identity[?, Section 7.12.2][?] when expressed in terms of polylogarithms; its derivation is reviewed in Appendix B. It is straightforward to invert the above and solve for ζ ; one gets

$$\zeta(1-s, x) = \frac{1}{2s} \left[e^{-i\pi s/2} \beta(x; s) + e^{i\pi s/2} \beta(1-x; s) \right]$$

thus proving the assertion that the Hurwitz zeta is an eigenfunction of the Bernoulli operator, with eigenvalue 2^{s-1} .

The eigenfunctions are smooth (infinitely differentiable) for the open interval $0 < x < 1$, but posses an essential singularity at $x = 0$, and more preciey, for all integer values of x . The singularity is of the form x^{1-s} as $x \rightarrow 0$. The eigenfunctions are square-integrable when $\Re s > 1$, although, depending on the value of s , the higher derivatives in general are not. The zeta functions have a simple pole at $s = 1$. The analytic properties are reviewed in Appendix C; a deeper, more complete picture of the analytic properties can be obtained by studying the polylogarithm[?].

These eigenfunctions are countably infinite degenerate at a given eigenvalue. Writing $s = \sigma + i\tau$ in terms of its real and imaginary components, any given eigenvalue takes

the form $\lambda = 2^{-s} = 2^{-\sigma} \exp(-i\tau \ln 2)$. Each such eigenvalue corresponds to a family of eigenvectors with $\tau' = \tau + 2\pi n / \ln 2$ for $n \in \mathbb{Z}$. One may take linear combinations of eigenvectors in this family. Clearly, any linear combination of only a finite number of such eigenvectors will also be smooth, and have the same general analytic properties.

In later sections, it will be seen the Bernoulli operator also has fractal eigenvectors. Some of these may be written as linear combinations of the zeta eigenvectors, although clearly these require the linear combination of an infinite number of the zetas. This begs several questions: are all linear combinations of the zetas that require an infinite series fractal? Or are there infinite linear combinations that are smooth? Is the set of smooth solutions connected or disconnected? Are the connected components simply connected or not? Answers to these questions are not immediately apparent.

7.2. The Kernel. What is the kernel of \mathcal{L}_B ? It is the set of functions that have only odd Fourier terms.

That is, for any integer $k \in \mathbb{Z}$ we have $\mathcal{L}_B \cos 2\pi(2k+1)x = 0$ and so we write $\cos 2\pi(2k+1)x \in K[\mathcal{L}_B]$ and likewise $\sin 2\pi(2k+1)x \in K[\mathcal{L}_B]$.

This implies that 'half' of all square-integrable functions are in the kernel. This is a huge space. The quotient space of the implied isomorphism thus has the Bernoulli polynomials as the representative elements. This is I think the correct way to relate coordinate space to the Hilbert space, is by means of the quotient space generated by the kernel of the time-evolution operator. XXX needs clarification XXX.

This begs more questions: is the quotient space simply connected, or not? Does this space have some non-trivial structure?

7.3. Synthetic operators. The equation 7.4 shows that there are a countable set of commuting operators all possessing the same eigenfunctions. Since they are commuting, one is free to take arbitrary sums and products. By applying series expansions and applying standard arguments from Fredholm operator theory, one may, for practical purposes consider analytic functions of these operators. These are "synthetic" operators, in that they have been synthesized from the basic set in equation 7.4. A few examples follow below.

Any Dirichlet series is straightforward: an operator with polylogarithm eigenvalues:

$$[\mathcal{L} \mathcal{I}_z] \beta(x; s) = \left[\sum_{p=1}^{\infty} z^p \mathcal{L}_p \right] \beta(x; s) = \sum_{p=1}^{\infty} \frac{z^p}{p^s} \beta(x; s) = \text{Li}_s(z) \beta(x; s)$$

An operator built from the Mobius function μ :

$$\left[\sum_{p=1}^{\infty} \mu(p) \mathcal{L}_p \right] \beta(x; s) = \sum_{p=1}^{\infty} \frac{\mu(p)}{p^s} \beta(x; s) = \frac{1}{\zeta(s)} \beta(x; s)$$

A curious squaring:

$$\left[\sum_{p=1}^{\infty} e^{2\pi i y p} \mathcal{L}_p \right] \beta(x; s) = \sum_{p=1}^{\infty} \frac{e^{2\pi i y p}}{p^s} \beta(x; s) = \frac{(2\pi)^s}{2\Gamma(s+1)} \beta(y; s) \beta(x; s)$$

The Euler products of Fredholm kernels may be taken; for example:

$$\prod_p \frac{1}{1 - \mathcal{L}_p} \beta(x; s) = \zeta(s) \beta(x; s)$$

with the product now taken only over prime numbers p . With considerable more difficulty, one might envision operators constructed after a sequence of differential and integral moves; for example, something involving the Mellin transform.

In principle, any such operator may be constructed. Denote $\beta(x; s)$ by the vector space basis vector e_s , and let e_s^* be its dual, so that $e_s^*(e_t) = \delta(s - t)$. Then given an arbitrary function $f(s)$, one may consider the operator constructed with the topological tensor product:

$$\mathcal{O} = \int_{\mathbb{C}} ds f(s) e_s \otimes e_s^*$$

then the operator \mathcal{O} has, by construction, eigenvectors $\beta(x; s)$ corresponding to eigenvalues $f(s)$. What makes the above different is that there is a certain air of concreteness to the p -adic transfer operators.

8. LATTICE MODEL TOPOLOGY

It is well-known that one-dimensional lattice models, and more generally, subshifts of finite type, can provide useful tools and insights for the study of one-dimensional iterated maps[?]. This section recaps the use of a one-dimensional spin lattice for the construction of a class of Bernoulli operator eigenfunctions.

Consider a one-sided lattice of points, labelled by the positive integers, that is, by $n \in \mathbb{N}$. At each lattice location, a variable σ_n , sometimes called the “spin”, can take one of p values, so that $\sigma_n \in \{0, 1, 2, \dots, p-1\}$. A given lattice configuration $\sigma = (\sigma_1, \sigma_2, \dots) \in \Omega$ can be taken to be either a p -adic number, or as a real number x , with the Cantor mapping $x : \Omega \rightarrow [0, 1]$ from the set of all lattice configurations Ω to the unit interval $[0, 1]$ given by

$$(8.1) \quad x(\sigma) = \sum_{n=1}^{\infty} \sigma_n p^{-n}$$

The latter mapping allows every possible spin lattice configuration to be mapped into the unit interval $[0, 1]$, although not vice-versa: the natural topology on the spin lattice is finer than the natural topology on the reals, in that $\sigma = (0, 1, 1, \dots)$ and $(1, 0, 0, \dots)$ are inequivalent lattice configurations, whereas the binary numbers $0.0111\dots$ and $0.1000\dots$ both represent the rational $1/2$. Only certain rationals are double-pointed like this, all irrationals are not. This double-pointed topology can be understood to be one and the same as the Cantor set topology, in that the Cantor set topology makes exactly the same distinction between endpoints of intervals[?]. For $p = 2$, the Stone representation theorem applies. Note also that lattices have another natural topology as well, the “cylinder set topology”, which is just the product topology of the discrete set of values at each lattice point.

This finer topology on the one-sided lattice allows a richer set of eigenfunctions to be constructed for the Bernoulli operator. One begins by noting that the shift operator τ on the lattice, defined by

$$\tau(\sigma) = \tau((\sigma_1, \sigma_2, \dots)) = (\sigma_2, \sigma_3, \dots)$$

corresponds exactly to the Bernoulli map:

$$x(\tau(\sigma)) = 2x(\sigma) - \lfloor 2x(\sigma) \rfloor$$

where we’ve written $x(\sigma)$ instead of x to help remind that x should be thought of as a map, not a real number. For the remainder of this section, this distinction will rarely be made again, although every occurrence of x should be implicitly understood to be the map $x : \Omega \rightarrow [0, 1]$.

Consider now some arbitrary function $V : \Omega \rightarrow F$, where F may be taken to be the field \mathbb{R} , or \mathbb{C} , or some arbitrary general field. Then, given some $\lambda \in F$, one may construct the

sum

$$(8.2) \quad H(\sigma) = \sum_{n=1}^{\infty} \lambda^n V(\tau^n \sigma)$$

where $\tau^n \sigma = (\tau \circ \tau \circ \dots \circ \tau)(\sigma)$ is simply the n -fold iteration of the shift operator. In physics, the function H is usually called the translation-invariant Hamiltonian for the system, while V is the interaction potential. For a one-sided lattice, H is not strictly translation invariant, as

$$(8.3) \quad H(\tau(\sigma)) = \sum_{n=1}^{\infty} \lambda^n V(\tau^{n+1} \sigma) = \frac{H(\sigma) - V(\sigma)}{\lambda}$$

so that H is “almost” an eigenvector of τ , with eigenvalue $1/\lambda$. Almost, because the one-sided lattice introduces a correction V/λ . For the bi-infinite, two-sided lattice, a truly translation-invariant Hamiltonian can be defined. The two-sided lattice model corresponds to the Bakers map, and is treated in greater detail in XXX.

If τ is the left-shift operator, then the analog of the transfer operator can be roughly taken to be the right-shift operator. Since the lattice is one-sided, there exist in fact p inequivalent right-shift operators S_k :

$$(8.4) \quad S_k(\sigma_1, \sigma_2, \dots) = (k, \sigma_1, \sigma_2, \dots)$$

These right-shift operators are inverses to the left shift operator τ only on one side, in that $\tau \circ S_k$ is the identity, but $S_k \circ \tau$ is not. Clearly, τ has no left-inverse, as τ throws away data in its action. For the $p = 2$ case, the shift operators act on the Cantor mapping as

$$(8.5) \quad \begin{aligned} x(S_0(\sigma)) &= \frac{x(\sigma)}{2} \\ x(S_1(\sigma)) &= \frac{1+x(\sigma)}{2} \end{aligned}$$

and so one may immediately recognize how to pose the Bernoulli operator on the lattice model. For some arbitrary $f : \Omega \rightarrow F$ one defines

$$[\mathcal{L}_B f](\sigma) = \frac{1}{2} [f(S_0(\sigma)) + f(S_1(\sigma))]$$

For the general p -adic case, this generalizes trivially:

Definition 8.1. The p -adic Bernoulli operator acting on the dual space of lattice configurations is given by

$$(8.6) \quad [\mathcal{L}_p f](\sigma) = \frac{1}{p} \sum_{k=0}^{p-1} f(S_k(\sigma))$$

where S_k is the k 'th right-shift operator.

The action of \mathcal{L}_p on the Hamiltonian H is takes the form

$$(8.7) \quad \mathcal{L}_p H = \mathcal{L}_p V + \lambda H$$

This is easily shown by direct substitution.

The critical result here is that H is an eigenvector provided that $\mathcal{L}_p V = 0$. Curiously, this implies that functions in the kernel of \mathcal{L}_B can be used to construct eigenfunctions of \mathcal{L}_B . The above construction is not limited the the Bernoulli operator, but clearly is a general result. One may construct a large number of general eigenfunctions, provided that on can find non-trivial objects in the kernel, and provided that one is working in a topology where the sum of equation 8.2 converges.

8.1. Fractal eigenfunctions. The above derivation of the eigenfunctions assumed a lattice model topology for the unit interval. Using equation 8.1 to regain the natural topology on the real numbers, one may use the construction to rediscover the full set of eigenfunctions of the Bernoulli operator. This is done explicitly in this section.

From equation 6.1, it was seen that the kernel of the Bernoulli operator consists of the odd harmonics:

$$\mathcal{L}_B \sin 2\pi(2k+1)x = 0$$

These correspond precisely to the zero-columns of equation 7.2. Inserting these into equation 8.2, one has

$$H(x) = \sum_{n=0}^{\infty} \lambda^n \sin(2\pi(2k+1)\tau^n x)$$

which is an eigenfunction: $\mathcal{L}_B H = \lambda H$. Under the sine, one may replace $\tau^n(x)$ by $2^n x$. There are also a set of eigenfunctions for the cosine. Combining all of these elements together, and changing notation, one has a collection of eigenfunctions

$$(8.8) \quad \phi_{\lambda,k}^B(x) = \sum_{n=0}^{\infty} \lambda^n \exp(2\pi i 2^n (2k+1)x)$$

for any $k \in \mathbb{Z}$. This derivation allowed $\lambda \in F$ for any field F ; taking F to be the complex numbers, the sum converges absolutely for $|\lambda| < 1$ and conditionally for $|\lambda| = 1$, depending on whether λ is a root of unity or not.

The superscript B denotes that these are eigenfunctions for the $p=2$ case of the Bernoulli operator. For the general case, one builds the eigenfunctions out of the shift states

$$v_{n,k,r}^{(p)}(x) = \exp 2\pi i x (pk+r) p^n$$

for $n \in \mathbb{N}_0$, $k \in \mathbb{Z}$ and r the ‘‘remainder’’, $0 < r < p$.

Theorem 8.2. *The functions $v_{n,k,r}^{(p)}(x)$ are shift states of \mathcal{L}_p , that is,*

$$(8.9) \quad \left[\mathcal{L}_p v_{n,k,r}^{(p)} \right] (x) = v_{n-1,k,r}^{(p)}(x)$$

for $n \neq 0$ and $\left[\mathcal{L}_p v_{0,k,r}^{(p)} \right] (x) = 0$ for the $n=0$ case.

Proof. This is obtained by simple substitution:

$$\begin{aligned} \left[\mathcal{L}_p v_{n,k,r}^{(p)} \right] (x) &= \frac{1}{p} \sum_{m=0}^{p-1} v_{n,k,r}^{(p)} \left(\frac{x+m}{p} \right) \\ &= \frac{1}{p} \exp [2\pi i x (pk+r) p^{n-1}] \sum_{m=0}^{p-1} \exp \left[2\pi i \frac{m}{p} (pk+r) p^n \right] \\ &= v_{n-1,k,r}^{(p)}(x) \end{aligned}$$

The sum in the middle line above is a Gauss sum, and may be shown to equal p when $n \neq 0$ and equal to zero otherwise. \square

The above relationship is reminiscent of the creation and annihilation ladder operators of quantum mechanics. This correspondance can be made stronger, and is explored in a later section. The p -adic shift states of equation 8.9 can be trivially used to construct the eigenstates of the p -adic Bernoulli operator. These are given by

$$\phi_{\lambda,k,r}^{(p)}(x) = \sum_{n=0}^{\infty} \lambda^n \exp(2\pi i x (pk+r) p^n)$$

which clearly obey

$$\mathcal{L}_p \phi_{\lambda,k,r}^{(p)} = \lambda \phi_{\lambda,k,r}^{(p)}$$

These eigenfunctions are highly degenerate, and span an infinite-dimensional space labelled by k and r . Any linear combination of these is also an eigenvector with the same eigenvalue. These clearly form a continuous spectrum for the Bernoulli operator, although, strictly speaking, this spectrum is defined only on the unit disk $|\lambda| < 1$. This is in contrast to the zeta eigenfunctions, which are defined for the entire complex λ -plane. This implies that the construction of eigenfunctions by means of lattice shifts is somehow inately limited: it allows some, but not all possible eigenfunctions to be found. The true nature of this obstruction is unclear. The proximal cause of the obstruction is that $\sum_n \lambda^n = 1/(1-\lambda)$ has a pole at $\lambda = 1$, thus preventing the convergence of the analytic series beyond that point. However, in the more global view, one has an operator acting on an infinite-dimensional topological vector space; whether the obstruction is due to the operator or the structure of the space itself is unclear.

8.2. Analytic properties. Unlike the zeta function eigenvectors, these eigenvectors are not infinitely differentiable. Taking the first derivative, one has

$$\frac{d}{dx} \phi_{\lambda,k,r}^{(p)}(x) = 2\pi i (pk+r) \sum_{n=0}^{\infty} p^n \lambda^n \exp(2\pi i x (pk+r) p^n)$$

and so the right hand side converges absolutely only if $|\lambda| < 1/p$. That is, for $|\lambda| > 1/p$, the eigenfunction is continuous but not differentiable. This is visualized in figure xxx. Smaller eigenvectors are differentiable n times only if $|\lambda| < 1/p^{n+1}$.

Clearly, any linear combination of only a finite number of these fractal eigenfunctions will posses similar analytic properties (these get mixed if different values of p are combined). However, there is an infinite series combination of these that reproduce the zeta function eigenfunctions. The explicit relationship between these two is developed in the next section.

8.3. The Fractal Spectrum. The fractal spectrum can be re-expressed in terms of the zeta eigenfunctions, and vice-versa. That is, the one set of eigenfunctions can be expressed as linear combinations of the other. This section gives the explicit linear relations connecting the one to the other.

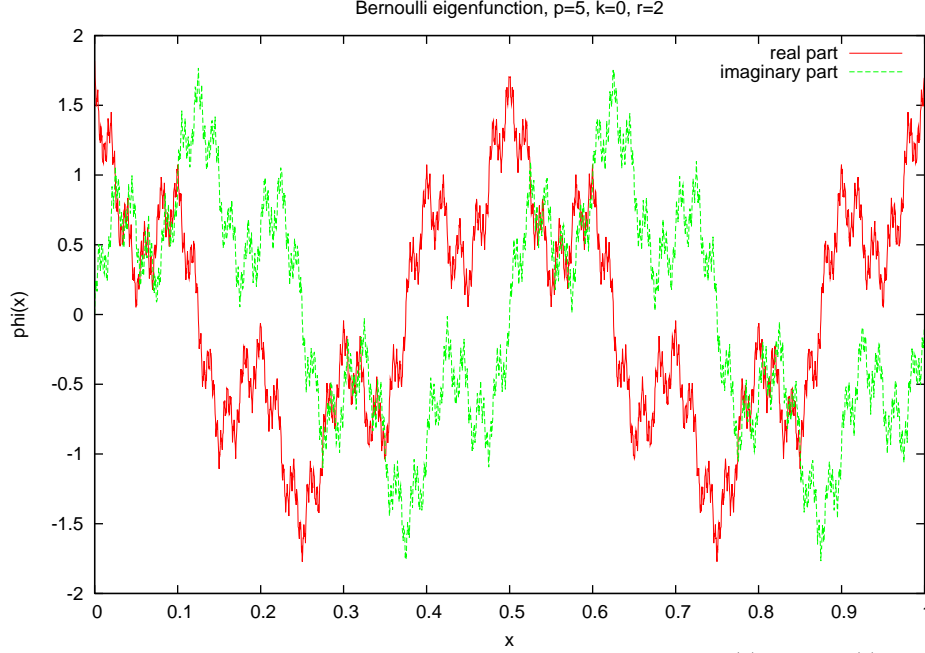
The establishment of this result is the observation that every integer has a unique factorization in the form of $(pk+r)p^n$, with k, r and n uniquely determined when p is fixed. This leads to a summation identity: for a fixed integer p , and some arbitrary summable function $f(n)$, one has

$$\sum_{n=0}^{\infty} f(n) = \sum_{k=0}^{\infty} \sum_{r=1}^{p-1} \sum_{m=0}^{\infty} f((pk+r)p^m)$$

The summand of the triple sum on the right is precisely of the form appearing in the fractal eigenfunctions, whereas the left-hand side is of the simpler form of the zeta eigenfunctions. The desired linear relationship is obtained merely by matching terms in the series. That is, one sets $f(n) = 2\Gamma(s+1) (2\pi n)^{-s} \exp(2\pi i x n)$ to immediately obtain

$$\begin{aligned} \beta(x; s) &= \frac{2\Gamma(s+1)}{(2\pi)^s} \sum_{k=0}^{\infty} \sum_{r=1}^{p-1} \frac{1}{(pk+r)^s} \sum_{m=0}^{\infty} \frac{\exp 2\pi i x (pk+r) p^m}{p^{ms}} \\ &= \frac{2\Gamma(s+1)}{(2\pi)^s} \sum_{k=0}^{\infty} \sum_{r=1}^{p-1} \frac{1}{(pk+r)^s} \phi_{\lambda,k,r}^{(p)}(x) \end{aligned}$$

FIGURE 8.1. Fractal Eigenfunction



This figure shows the real and imaginary parts of the eigenfunction $\phi_{\lambda,k,r}^{(p)}(x) = \phi_{0.45,0,2}^{(5)}(x)$. The non-differentiability is clearly visible. This function would be differentiable only for $\lambda < 0.2$; in that case, visually, the spikes would be completely smoothed away. The twice-repeated sine-wave nature is entirely due to $k = 2$.

where one identifies $\lambda = p^{-s}$.

The full set of zeta eigenfunctions, accounting for degeneracy, is given by

$$\beta_n(x; s) \equiv \beta(x; s + 2\pi ni / \ln 2)$$

with $n \in \mathbb{Z}$. These share the same eigenvalue: $\mathcal{L}_p \beta_n = p^{-s} \beta_n$. The relevant values of s should be restricted to a principal domain $-\pi < \Im s \ln 2 = \arg z < \pi$. Repeating the above manipulations, the goal is to write the zetas as a linear combination of the fractal eigenfunctions, so:

$$\beta_n(x; s) = \sum_{k=0}^{\infty} \sum_{r=1}^{p-1} F_{nkr} \phi_{\lambda,k,r}^{(p)}(x)$$

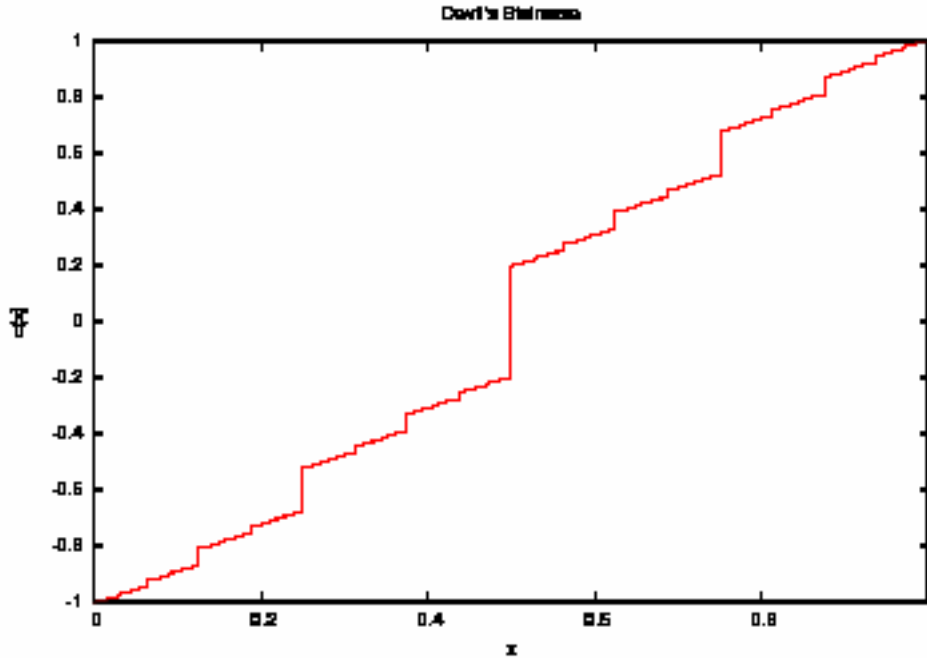
These matrix elements are

$$F_{nkr} = 2\Gamma\left(s + 1 + \frac{2\pi ni}{\ln 2}\right) (2\pi(pk+r))^{-s} \exp\left[-2n\pi i \frac{\ln \pi(pk+r)}{\ln 2}\right]$$

Presumably F is invertible.

The analytic situation is curious: the zeta eigenfunctions are well-defined for $|\lambda| > 1$, even as the fractal eigenfunctions are not. The change-of-basis summations clearly appear to be divergent as $|\lambda|$ approaches 1. It would appear that the zeta eigenfunctions span a larger space than the fractal eigenfunctions.

FIGURE 8.2. Devil's staircase



This figure shows a graph of the eigenfunction $c_\lambda(x)$ for $\lambda = 0.4$. As can be clearly seen, it takes on values in the Cantor set.

8.4. Some discontinuous eigenfunctions. The product topology suggests some additional eigenfunctions of the Bernoulli operator, eigenfunctions which are not continuous in the natural topology of the reals. Thus, for example, given the expansion in binary digits $b_n \in \{0, 1\}$ of a real number $0 \leq x \leq 1$,

$$x = \sum_{n=0}^{\infty} \frac{b_n}{2^{n+1}}$$

Then the Cantor or “Devil’s staircase” function

$$c_\lambda(x) = \lambda(\lambda - 1) \sum_{n=0}^{\infty} (-1)^{b_n} \lambda^n$$

is easily shown to be an eigenfunction of \mathcal{L}_2 with eigenvalue λ . This function is clearly discontinuous for all λ except $\lambda = 1/2$, where $c_\lambda(x) = B_1(x) = x - 1/2$. The Devil’s staircase function is shown in 8.4.

This construction can be generalized. The $\phi_{\lambda,k,r}^{(p)}(x)$ were built up explicitly from sine waves, but there is no particular reason to start with a sine-wave basis. Consider, for example, the square wave or Haar wavelet

$$h(x) = \begin{cases} +1 & \text{for } 0 \leq x < 1/2 \\ -1 & \text{for } 1/2 \leq x < 1 \end{cases}$$

extended to the entire real-number line as a periodic function $h(x+n) = h(x)$ for integer n . The Haar wavelet is in the kernel of the Bernoulli operator: $\mathcal{L}_2 h(x) = 0$, and thus can

be used to create a fractal eigenfunction by means of the lattice shift equation 8.7:

$$(8.10) \quad \psi_k(x) = \sum_{n=0}^{\infty} \lambda^n h((2k+1)2^n x)$$

so that $\mathcal{L}_2 \psi_k = \lambda \psi_k$ for any $|\lambda| < 1$ in the unit disk.

There are several aspects of this eigenfunction that are notable. Taken as a function on the real number line with the natural topology, it is discontinuous on a set dense in the reals: it is discontinuous at the dyadic rationals (that is, at multiples of 2^{-n}). By starting with the Haar wavelet, it also avoids the Gibbs phenomenon of “ringing” that one gets by working in a Fourier basis. Finally, the Haar wavelet is more compatible with the discrete, Cantor topology of the lattice; it has a particularly simple form when considered as a function on the lattice.

8.5. Fractal Symmetry and the Dyadic Monoid. The fractal eigenfunctions are self-similar. This self-similarity follows directly from the shift invariance of the lattice construction, that is, of the translation invariance of the lattice Hamiltonian embodied in equation 8.3. Loosely speaking, one might say that translation-invariant Hamiltonians must necessarily have fractal eigenfunctions. The self-similarities have a non-trivial structure, which is exposed in this section.

For concreteness, consider first the action of the right-shift operators S_0 and S_1 , defined by equation 8.4, acting on the $k = 0$ dyadic eigenfunction 8.10 built from the Haar wavelet. One has (dropping the subscript k from ψ):

$$\begin{aligned} [S_0 \cdot \psi](x) &= \psi\left(\frac{x}{2}\right) \\ &= \sum_{n=0}^{\infty} \lambda^n h(2^{n-1}x) \\ &= h\left(\frac{x}{2}\right) + \lambda \sum_{n=1}^{\infty} \lambda^{n-1} h(2^{n-1}x) \\ &= h\left(\frac{x}{2}\right) + \lambda \psi(x) \\ &= -1 + \lambda \psi(x) \end{aligned}$$

with the last step following because the Haar wavelet is identically -1 on the interval $0 \leq x < 1/2$. To emphasize this, write it more simply as

$$\psi\left(\frac{x}{2}\right) = -1 + \lambda \psi(x)$$

That is, one very explicitly has that the eigenfunction on the whole unit interval looks exactly like the eigenfunction on the half-interval, except that its scaled and offset: this is classic fractal self-similarity.

For the other shift operator S_1 , one readily obtains that $S_1 \psi = 1 + \lambda \psi$. The action of these two shift operators has a representation in terms of 2×2 matrices acting on two-dimensional vectors. The basis vectors e_0 and e_1 for this vector space are

$$1 \mapsto e_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \psi \mapsto e_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

one has

$$(8.11) \quad S_0 = \begin{bmatrix} 1 & -1 \\ 0 & \lambda \end{bmatrix} \quad \text{and} \quad S_1 = \begin{bmatrix} 1 & 1 \\ 0 & \lambda \end{bmatrix}$$

so that one has, for example $S_0 e_1 = -e_0 + \lambda e_1$ as the action of the right shift on this vector space. The two matrices S_0 and S_1 generate a monoid, the *dyadic monoid*. A general element γ of the monoid is then

$$\gamma = S_0^{n_0} S_1^{n_1} S_0^{n_2} S_1^{n_3} \dots$$

for non-negative integers n_0, n_1, n_2, \dots . The monoid is free, in that there are no relations between S_0 and S_1 , and each unique string of integers n_0, n_1, n_2, \dots results in a unique element of the monoid. For the purpose of this text, a *monoid* is defined much as a group, except that inverses are not considered. This is for two reasons: first, the inverses of S_0 and S_1 are not uniquely defined: the left and right inverse differ. Secondly, γ is to be understood to be the action of a monoid element acting on the unit interval. As long as the action maps the whole interval into a sub-interval, the action is well-defined; however, there are technical difficulties with going in the reverse direction.

The dyadic monoid has many interesting properties; these are discussed in greater detail in [?]. Among the notable properties is that the dyadic monoid is isomorphic to a certain subset of the Modular Group; this connection alone being important as the modular group plays an important role in number theory, specifically in the area of modular forms and automorphic forms. Not inconsequential is that the theory of modular forms again borders on the theory of the Riemann zeta. In a similar vein, one notes that the string of integers $[n_0, n_1, n_2, \dots]$ is isomorphic to a continued fraction. Continued fractions are important both to the theory of Diophantine equations (Pellian equations), and in turn are seen to tie back into general phenomena seen in ergodic dynamics.

The integral of the Haar wavelet or square wave is the tent map or triangle wave

$$t(x) = \int_0^x h(y) dy = \begin{cases} \frac{1}{2} - 2x & \text{when } 0 \leq x \leq 1/2 \\ -\frac{3}{2} + 2x & \text{when } 1/2 \leq x \leq 1 \end{cases}$$

This is extended to the entire real line by defining $t(x+n) = t(x)$ for integer n . The triangle wave has the curious property of period-doubling upon iteration: that is, $t^k(x) = -t(2^{k-1}x)$ for $k > 1$. The iterated tent map behaves as a shift state, in that the Bernoulli operator acts as an annihilation operator:

$$[\mathcal{L}_B t^k](x) = t^{k-1}(x)$$

with the shift terminating at $k = 1$:

$$[\mathcal{L}_B t](x) = 0$$

The fractal eigenfunction constructed from the triangle wave is the Takagi or blancmange curve:

$$b_\lambda(x) = \sum_{n=0}^{\infty} \lambda^n t(2^n x)$$

and so of course, $\mathcal{L}_B b_\lambda = \lambda b_\lambda$. There is a curious special case: the Takagi curve $b_{1/4}(x)$ is a parabola, corresponding to the Bernoulli polynomial $B_2(x)$. Apparently, the construction of the parabola by self-similar subdivision was known to Archimedes[?].

The blancmange curve transforms under a three-dimensional representation as the Haar-wavelet curve, in that

$$S_0 b_\lambda = -\left(x - \frac{1}{2}\right) + \lambda b_\lambda \quad \text{and} \quad S_1 b_\lambda = \left(x - \frac{1}{2}\right) + \lambda b_\lambda$$

Taking as the basis vectors

$$\begin{aligned} 1 \mapsto e_0 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ x \mapsto e_1 &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ b_\lambda \mapsto e_2 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

the two generators may be written as

$$S_0 = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \lambda \end{bmatrix} \quad \text{and} \quad S_1 = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \lambda \end{bmatrix}$$

This process can be extended arbitrarily. One may choose to integrate the triangle wave, or to make a more general ansatz. Given some arbitrary polynomial $p(x)$, one may fashion a piece-wise polynomial periodic wave

$$q(x) = \begin{cases} -p(x) & \text{when } 0 \leq x < 1/2 \\ p(x - \frac{1}{2}) & \text{when } 1/2 \leq x < 1 \end{cases}$$

so that $q(x + 1/2) = -q(x)$ or equivalently, $[\mathcal{L}_B q](x) = 0$. From this, one constructs the fractal wave

$$f_\lambda(x) = \sum_{n=0}^{\infty} \lambda^n q(2^n x)$$

which transforms as

$$S_0 f_\lambda = -p\left(\frac{x}{2}\right) + \lambda f_\lambda \quad \text{and} \quad S_1 f_\lambda = p\left(\frac{x}{2}\right) + \lambda f_\lambda$$

If the degree of the polynomial is m , then the associated linear representation is $m + 2$ dimensional. One may choose as the basis vectors $e_k = x^k$ for $k \leq m$ and $e_{m+1} = f_\lambda$. The matrix forms of S_0 and S_1 are obtained by expanding the polynomial $p(x/2)$ in terms of the chosen basis. There is nothing special about the monomial basis; one might consider any basis where e_k represents a polynomial of degree k .

As should be clear from the above construction, when $q(x)$ is not a polynomial, the representation cannot be finite-dimensional. In particular, the sine-wave based dyadic fractal 8.8 constructed from $q(x) = \exp 2\pi i x$ does not transform under a finite-dimensional representation.

The general dyadic case $k \neq 0$, given by

$$f_{\lambda,k}(x) = \sum_{n=0}^{\infty} \lambda^n q(2^n (2k+1)x)$$

does not change the situation. Since $[\mathcal{L}_B q](x) = 0$, one has $q(x + 1/2) = -q(x)$, and so again, the action of S_0 and S_1 is given by

$$S_0 f_{\lambda,k} = q\left((2k+1)\frac{x}{2}\right) + \lambda f_{\lambda,k} \quad \text{and} \quad S_1 f_{\lambda,k} = -q\left((2k+1)\frac{x}{2}\right) + \lambda f_{\lambda,k}$$

As before, if $q(x)$ is made from a polynomial of degree m , the resulting fractal function transforms under an $m + 2$ dimensional linear representation.

The general p -adic case proceeds in a similar manner. Polynomials of degree m transform as an $m + 2$ dimensional representation, just as before. There are no longer two generators, but p generators S_0, S_1, \dots, S_{p-1} .

XXXX Would deeper properties of the dyadic monoid be worth reviewing here???

XXXX – When are the inf-dimensional reps conjugate? What are conjugacy classes? etc.

XXXX more info on the p -adic free monoids, please.

8.6. Number theoretic connections. XXX The definition 8.12 is suggestive of the kinds of sums occurring in the definition of the Hecke operator. One may also consider a Gauss-sum-like extension

$$(8.12) \quad \left[\mathcal{L}_B^{(m)} f \right] (\sigma) = \frac{1}{p} \sum_{k=0}^{p-1} e^{2\pi i k m / p} f(S_k(\sigma))$$

regaining $\mathcal{L}_B = \mathcal{L}_B^{(0)}$.

9. TWO-SIDED LATTICE

The two-sided lattice is the lattice that stretches off to both the left and the right. The state of the two-sided lattice may be represented by two numbers $0 \leq x, y \leq 1$ with x given by equation ??, and y likewise, but instead being a sum over the negative indexes, starting at $k = -1$. The energy is a simple sum of the energy of the half-lattices to the left and to the right, plus an interaction term involving spins on both the left and right. The probability density is a product of the probability densities on the left and right, adjusted by the interaction between the left and right sides.

The shift operator τ can be understood to be an operator that pops a bit off the right-hand half lattice, and pushes it onto the left half-lattice. In the dyadic notation, it can be recognized as the so-called *Baker's map*, a map from the unit square onto itself that cuts the square into two pieces, and stacks and squashes them. That is, given a pair of numbers (x, y) in the unit square, one has

$$(9.1) \quad \tau(x, y) = \left(\frac{x + \lfloor 2y \rfloor}{2}, 2y - \lfloor 2y \rfloor \right)$$

The Baker's map can be studied in its own right as a discrete-time dynamical system, in that τ can be taken to define the time evolution of a unit square over a single time-step. The time evolution is invertible, in that the inverse map τ^{-1} is uniquely defined, being

$$(9.2) \quad \tau^{-1}(x, y) = \left(2x - \lfloor 2x \rfloor, \frac{y + \lfloor 2x \rfloor}{2} \right)$$

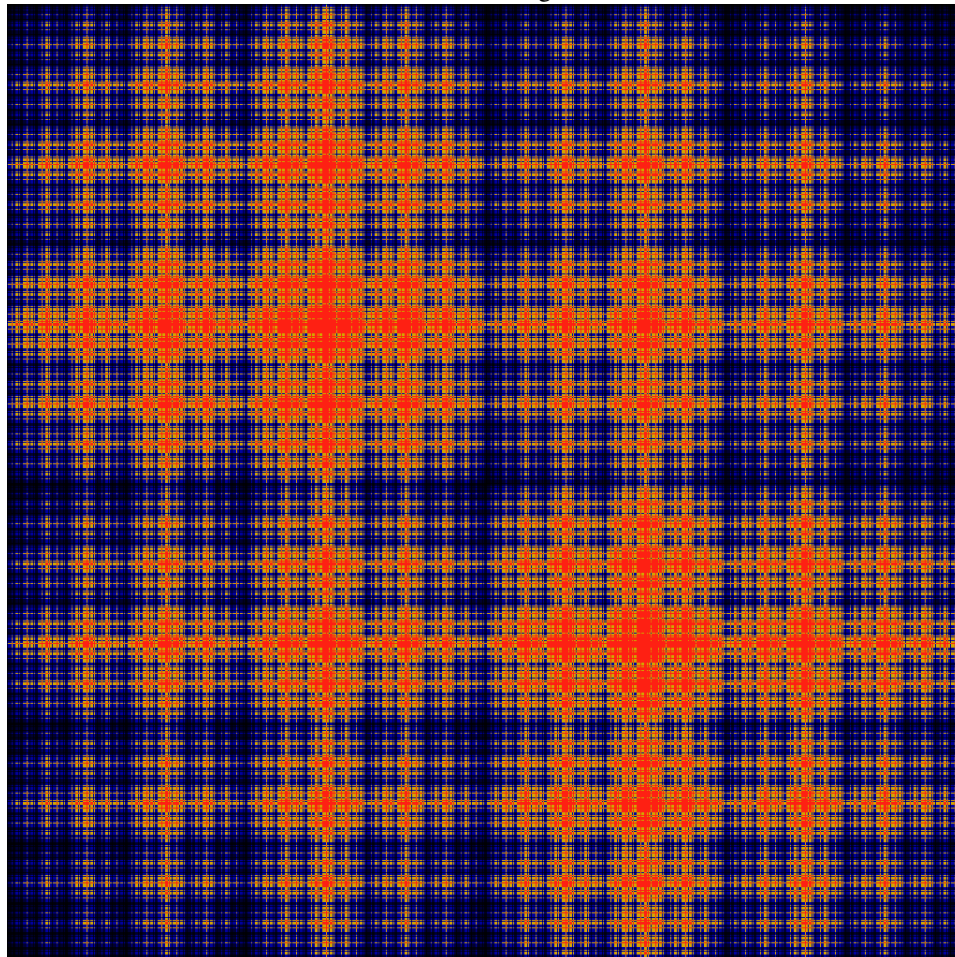
The inverse map can be seen to be identical to the forward map with x and y interchanged: that is, translation to the left on the lattice is identical to translation to the right, with left and right exchanged. One may say the map is PT -symmetric, with P standing for the “parity exchange” operation $x \leftrightarrow y$ and T being the “time inversion” operator $\tau \leftrightarrow \tau^{-1}$.

The two-sided lattice may be studied either as a lattice model, or as a dynamical system with time evolution given by τ . It is interesting to juxtapose these two viewpoints. As a lattice model, one is typically only interested in lattices for which the energy and the probability density are invariant under the action of the translation operator. That is, one is interested only in the classical Hamiltonians of eqn ?? which are translation-invariant:

$$(9.3) \quad H(\tau^n \sigma) = H(\sigma)$$

for all $n \in \mathbb{Z}$.

FIGURE 9.1. Ising Tartan



This tartan-like graph shows the Ising model probability density $P(\sigma)$ for the two-sided lattice using the dyadic mapping. That is, the lattice configuration $\sigma = (\sigma_{-N-1}, \dots, \sigma_{-2}, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \dots, \sigma_N)$ is represented by two numbers $0 \leq x, y \leq 1$ with

$$x(\sigma) = \sum_{k=0}^N \left(\frac{\sigma_k + 1}{2} \right) 2^{-(k+1)}$$

and

$$y(\sigma) = \sum_{k=0}^N \left(\frac{\sigma_{-k-1} + 1}{2} \right) 2^{-(k+1)}$$

The energy of a given configuration σ is computed using ??, with the sum running from $-N$ to $+N$, of course. The probability density $P(\sigma)$ is given by equation ??.

The graph here assumes the Ising potential, with $J = 0.3$ and $M = 0$ for a finite sized lattice with $N = 10$. The color choices here are such that black represents values where $P(\sigma) = P(x, y)$ are zero, blue are small values, with yellow and red being progressively larger values. This fractal tartan is invariant under the Baker's map.

Considered as a dynamical system, one is typically interested in the time evolution of densities on the unit square, that is, of real-valued maps $\rho : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$. The time evolution of a density can be simply understood as a physical model, where the density is a local density of some “dust” of points (x, y) , with the time evolution of each point given by τ . Thus, the time evolution of this “dust” or density is given by the transfer operator \mathcal{L}_T as

$$(9.4) \quad [\mathcal{L}_T \rho](x, y) = (\rho \circ \tau^{-1})(x, y) = \rho \left(2x - [2x], \frac{y + [2x]}{2} \right)$$

The transfer operator is thus a map $\mathcal{L}_T : \mathcal{F} \rightarrow \mathcal{F}$ where \mathcal{F} is the set of functions on the unit square. Clearly \mathcal{L}_T is a linear operator. Here, the subscript T on \mathcal{L}_T is used to remind us that this is now the transfer operator on the two-sided lattice; it can be written as a combination of \mathcal{L}_B and the Koopman operator \mathcal{K}_B for the single-sided lattice. One is interested in characterizing the eigenfunctions and eigenvalues of \mathcal{L}_T .

The general idea behind equation ?? can be immediately appealed to, to construct some of the eigenstates of \mathcal{L}_T . That is, one considers functions $f(x, y)$ of the form

$$(9.5) \quad f = \sum_{n=-\infty}^{\infty} \lambda^n g \circ \tau^n$$

given some function $g(x, y)$ and number λ . Formally, such an f is an eigenstate of \mathcal{L}_T with eigenvalue λ : that is, $\mathcal{L}_T f = \lambda f$. In practical terms, it is evident that not all possible λ can make the sum convergent, although one might expect that λ on the unit circle $|\lambda| = 1$ of the complex plane might lead to a convergent sum. Thus immediately, one deduces that \mathcal{L}_T must surely be a unitary operator. The unitarity of \mathcal{L}_T makes intuitive sense in a certain way: the map τ is invertible, and one expects time-reversible evolution to be described by unitary operators. The figure 9.1 shows a tartan-like distribution corresponding to $\lambda = 1$ and $g(x, y) = V_{\text{Ising}}(x)$ the Ising model potential. Some additional tartans, for other values of λ , are shown in figure xx.

The proper analysis of the transfer operator \mathcal{L}_T requires that the function space \mathcal{F} be pinned down more precisely. Classic results on time-symmetry breaking [?, ?, ?] indicate that whether or not \mathcal{L}_T is unitary depends on the function space \mathcal{F} .

9.1. Ladder operators. Simple Harmonic Oscillator review.

There are several related, intertwined confusions, here. There lattice allows two different interpretations of what it means to be a ladder operator. XXX finish me.

10. MEASURE-THEORETIC DESCRIPTION

In this section, it will be shown that the transfer operator is the push-forward of the shift operator; a theorem and a sequence of lemmas will be posed, that hold in general form. The point of this theorem is to disentangle the role of topology, and specifically, the role of measure theory, from the use of the shift operator. We begin with a general setting.

Consider a topological space X , and a field F over the reals \mathbb{R} . Here, F may be taken to be \mathbb{R} itself, or \mathbb{C} or some more general field over \mathbb{R} . The restriction of F to being a field over the reals is required, so that it can be used in conjunction with a measure; measures are always real-valued.

One may then define the algebra of functions $\mathcal{F}(X)$ on X as the set of functions $f \in \mathcal{F}(X)$ such that $f : X \rightarrow F$. An algebra is a vector space endowed with multiplication between vectors. The space $\mathcal{F}(X)$ is a vector space, in that given two functions $f_1, f_2 \in \mathcal{F}(X)$, their linear combination $af_1 + bf_2$ is also an element of $\mathcal{F}(X)$; thus f_1

and f_2 may be interpreted to be the vectors of a vector space. Multiplication is the point-wise multiplication of function values; that is, the product $f_1 f_2$ is defined as the function $(f_1 f_2)(x) = f_1(x) \cdot f_2(x)$, and so $f_1 f_2$ is again an element of $\mathcal{F}(X)$. Since one clearly has $f_1 f_2 = f_2 f_1$, multiplication is commutative, and so $\mathcal{F}(X)$ is also a commutative ring.

The space $\mathcal{F}(X)$ may be endowed with a topology. The coarsest topology on $\mathcal{F}(X)$ is the *weak topology*, which is obtained by taking $\mathcal{F}(X)$ to be the space that is topological dual to X . As a vector space, $\mathcal{F}(X)$ may be endowed with a norm $\|f\|$. For example, one may take the norm to be the L^p -norm

$$\|f\|_p = \left(\int |f(x)|^p dx \right)^{1/p}$$

For $p = 2$, this norm converts the space $\mathcal{F}(X)$ into the Hilbert space of square-integrable functions on X . Other norms are possible, in which case $\mathcal{F}(X)$ has the structure of a Banach space rather than a Hilbert space.

Consider now a homomorphism of topological spaces $g : X \rightarrow Y$. This homomorphism induces the pullback $g^* : \mathcal{F}(Y) \rightarrow \mathcal{F}(X)$ on the algebra of functions, by mapping $f \mapsto g^*(f) = f \circ g$ so that $f \circ g : Y \rightarrow F$. The pullback is a linear operator, in that

$$g^*(af_1 + bf_2) = ag^*(f_1) + bg^*(f_2)$$

That the pullback is linear is easily demonstrated by considering how g^*f acts at a point: $(g^*f)(x) = (f \circ g)(x) = f(g(x))$ and so the linearity of g^* on $af_1 + bf_2$ follows trivially.

One may construct an analogous mapping, but going in the opposite direction, called the push-forward: $g_* : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$. There are two ways of defining a push-forward. One way is to define it in terms of the sheaves of functions on subsets of X and Y . The sheaf-theoretic description is more or less insensitive to the ideas of measurability, whereas this is important to the definition of the transfer operator, as witnessed by the appearance of the Jacobian determinant in equation 2.2. By contrast, the measure-theoretic push-forward captures this desirable aspect. It may be defined as follows.

One endows the spaces X and Y with sigma-algebras (X, \mathcal{A}) and (Y, \mathcal{B}) , so that \mathcal{A} is the set of subsets of X obeying the axioms of a sigma-algebra, and similarly for \mathcal{B} . A mapping $g : X \rightarrow Y$ is called “measurable” if, for all Borel sets $B \in \mathcal{B}$, one has the pre-image $g^{-1}(B) \in \mathcal{A}$ being a Borel set as well. Thus, a measurable mapping induces a push-forward on the sigma-algebras: that is, one has a push-forward $g_* : \mathcal{F}(\mathcal{A}) \rightarrow \mathcal{F}(\mathcal{B})$ given by $f \mapsto g_*(f) = f \circ g^{-1}$, which is defined by virtue of the measurability of g . The push-forward is a linear operator, in that

$$g_*(af_1 + bf_2) = ag_*(f_1) + bg_*(f_2)$$

One regains the transfer operator as defined in equation 2.2 by considering the limiting behavior of the push-forward on progressively smaller sets. That is, one has

Theorem 10.1. *The transfer operator is the point-set topology limit of the measure-theoretic push-forward.*

Proof. The proof that follows is informal, so as to keep it simple. It is aimed mostly at articulating the language and terminology of measure theory. The result is none-the-less rigorous, if taken within the confines of the definitions presented.

Introduce a measure $\mu : \mathcal{A} \rightarrow \mathbb{R}^+$ and analogously $\nu : \mathcal{B} \rightarrow \mathbb{R}^+$. The mapping g is measure-preserving if ν is a push-forward of μ , that is, if $\nu = g_*\mu = \mu \circ g^{-1}$. The measure is used to rigorously define integration on X and Y . Elements of $\mathcal{F}(\mathcal{A})$ can be informally

understood to be integrals, in that $f(A)$ for $A \in \mathcal{A}$ may be understood as

$$f(A) = \int_A \tilde{f}(z) d\mu(z) = \int_A \tilde{f}(z) |\mu'(z)| dz$$

where $|\mu'(x)|$ is to be understood as the Jacobean determinant at a point $x \in X$. Here, \tilde{f} can be understood to be a function that is being integrated over the set A , whose integral is denoted by $f(A)$. The value of \tilde{f} at a point $x \in X$ can be obtained by means of a limit. One considers a sequence of $A \in \mathcal{A}$, each successively smaller than the last, each containing the point x . One then has

$$\lim_{A \ni x} \frac{f(A)}{\mu(A)} = \tilde{f}(x)$$

which can be intuitively proved by considering A so small that \tilde{f} is approximately constant over A :

$$f(A) = \int_A \tilde{f}(z) d\mu(z) \approx \tilde{f}(x) \int_A d\mu = \tilde{f}(x) \mu(A)$$

To perform the analogous limit for the push-forward, one must consider a point $y \in Y$ and sets $B \in \mathcal{B}$ containing y . In what follows, it is now assumed that $g : X \rightarrow Y$ is a multi-sheeted countable covering of Y by X . By this it is meant that for any y that is not a branch-point, there is a nice neighborhood of y such that its pre-image consists of the union of an at most countable number of pair-wise disjoint sets. That is, for y not a branch point, and for $B \ni y$ sufficiently small, one may write

$$g^{-1}(B) = A_1 \cup A_2 \cup \dots = \bigcup_{j=1}^k A_j$$

where k is either finite or stands for ∞ , and where $A_i \cap A_j = \emptyset$ for all $i \neq j$. At branch points, such a decomposition may not be possible. The axiom of sigma-additivity guarantees that such multi-sheeted covers behave just the way one expects integrals to behave: in other words, one has

$$\mu(g^{-1}(B)) = \mu\left(\bigcup_{j=1}^k A_j\right) = \sum_{j=1}^k \mu(A_j)$$

whenever the collection of A_j are pair-wise disjoint. Similarly, in order to have the elements $f \in \mathcal{F}(\mathcal{A})$ behave as one expects integrals to behave, one must restrict $\mathcal{F}(\mathcal{A})$ to contain only sigma-additive functions as well, so that

$$f(g^{-1}(B)) = f\left(\bigcup_{j=1}^k A_j\right) = \sum_{j=1}^k f(A_j)$$

As the set B is taken to be smaller and smaller, the sets A_j will become smaller as well. Denote by x_j the corresponding limit point of each A_j , so that $g(x_j) = y$ and the pre-image of y consists of these points: $g^{-1}(y) = \{x_1, x_2, \dots \mid g(x_j) = y\}$. One now combines these

provisions to write

$$\begin{aligned}
[g_*\tilde{f}](y) &= \lim_{B \ni y} \left[\frac{(g_*f)(B)}{\nu(B)} \right] \\
&= \lim_{B \ni y} \left[\frac{(f \circ g^{-1})(B)}{\nu(B)} \right] \\
&= \lim_{A_j \ni g^{-1}(y)} \frac{f(A_1 \cup A_2 \cup \dots)}{\nu(B)} \\
&= \lim_{A_j \ni g^{-1}(y)} \frac{\sum_{j=1}^k f(A_j)}{\nu(B)} \\
(10.1) \qquad &= \sum_{j=1}^k \tilde{f}(x_j) \lim_{A_j \ni x_j} \frac{\mu(A_j)}{\nu(B)}
\end{aligned}$$

The limit in the last line of this sequence of manipulations may be interpreted in two ways, depending on whether one wants to define the measure ν on Y to be the push-forward of μ , or not. If one does take it to be the push-forward, so that $\nu = g_*\mu$, then one has

$$\lim_{A_j \ni x_j} \frac{\mu(A_j)}{g_*\mu(B)} = \frac{1}{|g'(x_j)|}$$

where $|g'(x_j)|$ is the Jacobian determinant of g at x_j . This last is a standard result of measure theory, and can be intuitively proved by noting that $g(A_j) = B$, so that

$$\nu(B) = \int_{A_j} g'(z) d\mu(z) \approx g'(x_j) \mu(A_j)$$

for “small enough” B . Assembling this with the previous result, one has

$$(10.2) \qquad [g_*\tilde{f}](y) = \sum_{x_j \in g^{-1}(y)} \frac{\tilde{f}(x_j)}{|g'(x_j)|}$$

which may be easily recognized as equation 2.2. This concludes the proof of the theorem, that the transfer operator is just the point-set topology limit of the push-forward. \square

In simplistic terms, the push-forward can be thought of as a kind of change-of-variable. Thus, one should not be surprised by the following lemma, which should be recognizable as the Jacobian, from basic calculus.

Lemma 10.2. (Jacobian) *One has*

$$\sum_{x_j \in g^{-1}(y)} \frac{1}{|g'(x_j)|} = 1$$

Proof. This follows by taking the limit $\overrightarrow{A_j \ni x_j}$ of

$$\frac{\mu(A_j)}{g_*\mu(B)} = \frac{\mu(A_j)}{\sum_{i=1}^k \mu(A_i)}$$

and then summing over j . \square

Corollary 10.3. *The uniform distribution (i.e. the measure) is an eigenvector of the transfer operator, associated with the eigenvalue one.*

Proof. This may be proved in two ways. From the viewpoint of point-sets, one simply takes $\tilde{f} = \text{const.}$ in equation 10.2, and applies the lemma above. From the viewpoint of the sigma-algebra, this is nothing more than a rephrasing of the starting point, that $\nu = g_*\mu$, and then taking the space $Y = X$, so that the push-forward induced by $g : X \rightarrow X$ is a measure-preserving map: $g_*\mu = \mu$. \square

The last corollary is more enlightening when it is turned on its side; it implies two well-known theorems, which follow easily in this framework.

Corollary 10.4. (*Ruelle-Perron-Frobenius theorem*). *All transfer operators are continuous, compact, bounded operators; furthermore, they are isometries of Banach spaces.*

Proof. This theorem is of course just the Frobenius-Perron theorem, recast in the context of measure theory. By definition, the measures have unit norm: that is, $\|\mu\|_1 = 1$ and $\|\nu\|_1 = 1$. This is nothing more than the statement that the spaces X and Y are measurable: the total volume of X and Y is, by definition, one. Since $\nu = g_*\mu$, we have $\|g_*\mu\|_1 = 1$, and this holds for all possible measures $\mu \in \mathcal{F}(X)$.

Recall the definition of a bounded operator. Given a linear map $T : U \rightarrow V$ between Banach spaces U and V , then T is bounded if there exists a constant $C < \infty$ such that $\|Tu\|_V \leq C\|u\|_U$ for all $u \in U$. But this is exactly the case above, with $T = g_*$, and $U = \mathcal{F}(X)$, $V = \mathcal{F}(Y)$, and $C = 1$. The norm of a bounded operator is conventionally defined as

$$\|T\| = \sup_{u \neq 0} \frac{\|Tu\|_V}{\|u\|_U} = \sup_{\|u\|_U \leq 1} \|Tu\|_V$$

and so we have the norm of g_* being $\|g_*\| = 1$. That g_* is an isometry follows trivially from $\|g_*\mu\|_1 = \|\nu\|_1$ and that g_* is linear. \square

The corollary 10.3 can also be treated as a corollary to the Perron-Frobenius theorem: namely, that there is at least one vector that corresponds to the maximum eigenvalue of g_* . This eigenvector is in fact the Haar measure, as the next theorem shows.

Theorem 10.5. (*Haar measure*) *For any homomorphism $g : X \rightarrow X$, one may find a measure μ such that $g_*\mu = \mu$; that is, every homomorphism g of X induces a measure μ on X such that g is a measure-preserving map. If g is ergodic, then the measure is unique.*

Proof. By definition, μ is a fixed point of g_* . The fixed point exists because g_* is a bounded operator, and the space of measures is compact, and so a bounded operator on a compact space will have a fixed point. The existence of the fixed point is given by the Markov-Kakutani theorem[?, p 456]. The Markov-Kakutani theorem also provides the uniqueness condition: if there are other push-forwards h_* that commute with g_* , then each such push-forward will also have a fixed point. The goal is then to show that when g is ergodic, there are no other functions h that commute with g . But this follows from the definition of ergodicity: when g is ergodic, there are no invariant subspaces, and the orbit of g is the whole space. As there are no invariant subspaces, there are no operators that can map between these subspaces, *i.e.* there are no other commuting operators. \square

A peculiar special case is worth mentioning: if g is not ergodic on the whole space, then typically one has that the orbit of g splits or foliates the measure space into a bunch of pairwise disjoint leaves, with g being ergodic on each leaf. The Markov-Kakutani theorem then implies that there is a distinct fixed point μ in each leaf, and that there is a mapping that takes μ in one leaf to that in another.

In the language of dynamical systems, the push-forward g_* is commonly written as \mathcal{L}_g , so that one has

$$g_* = \mathcal{L}_g : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$$

now being called the transfer operator or the Ruelle-Frobenius-Perron operator.

In the language of physics, the fixed point μ is called the “ground state” of a system. When it is unique, then the ground state is not degenerate; when it is not unique, then the ground state is said to be degenerate. The operator g_* is the time-evolution operator of the system; it shows how physical fields $f \in \mathcal{F}(X)$ over a space X evolve over time. When F is the complex numbers \mathbb{C} , the fact that $\|g_*\| = 1$ is essentially a way of stating that time-evolution is unitary; the Frobenius-Perron operator is the unitary time-evolution operator of the system. What is called “second quantization” in physics should be interpreted as the fitting of the space $\mathcal{F}(X)$ with a set of basis vectors, together with a formulation of g_* in terms of that basis.

11. THE TOPOLOGICAL ZETA

XXX ToDo: explain topo zeta gets this has this name, and why this concept is important.
XXX

The topological zeta of the Bernoulli operator can be computed very easily in the polynomial basis because we know the eigenvalues and these form a simple series. We’ll define the Bernoulli topological zeta as

$$(11.1) \quad \zeta_B(t) \equiv \frac{1}{\det[\mathbb{I} - t\mathcal{L}_B]}$$

We start by noting its inverse:

$$\begin{aligned} \det[\mathbb{I} - t\mathcal{L}_B] &= \prod_{n=0}^{\infty} (1 - t 2^{-n}) \\ &= 1 - t \sum_{j=0}^{\infty} 2^{-j} + t^2 \sum_{j=0}^{\infty} 2^{-j} \sum_{\substack{k=0 \\ k \neq j}}^{\infty} 2^{-k} - t^3 \dots \\ (11.2) \quad &= 1 - 2t + \frac{8}{3}t^2 - \frac{16}{7}t^3 + \frac{128}{105}t^4 - \dots \end{aligned}$$

Successive terms of this series are hard to compute, and it would be interesting to know what the generating function for this series is. The series appears to have a circle of convergence of radius one. The zeta can be computed directly by working with its logarithm:

$$\begin{aligned} \log \zeta_B(t) &= \log \prod_{n=0}^{\infty} (1 - t 2^{-n})^{-1} \\ &= - \sum_{n=0}^{\infty} \log(1 - t 2^{-n}) \\ &= \sum_{k=1}^{\infty} \frac{t^k}{k} \frac{2^k}{2^k - 1} \\ (11.3) \quad &= -\log(1-t) + \sum_{k=1}^{\infty} \frac{t^k}{k} \frac{1}{2^k - 1} \end{aligned}$$

Thus we have $\text{Tr} \mathcal{L}_B^k = 2^k / (2^k - 1)$. Of some curiosity is the proximity of the Erdos-Borwein constant:

$$(11.4) \quad \begin{aligned} 1.6066\dots &= \sum_{n=1}^{\infty} \frac{1}{2^n - 1} \\ &= \sum_{n=1}^{\infty} \frac{d(n)}{2^n} \end{aligned}$$

which marks the first appearance of a classical number-theoretic function in the proceedings so far: $d(n)$ is the number of divisors of n . This arises from the Lambert series

$$(11.5) \quad \sum_{n=1}^{\infty} d(n)x^n = \sum_{n=1}^{\infty} \frac{x^n}{1-x^n}$$

The sum

$$(11.6) \quad E(t) = \sum_{k=1}^{\infty} \frac{t^k}{1-2^{-k}}$$

can be re-summed as a Lambert series, namely,

$$(11.7) \quad E(t) = \sum_{k=1}^{\infty} b_k 2^{-k}$$

where

$$(11.8) \quad b_k = \sum_{n|k} (2t)^n$$

The analytic/meromorphic structure of this zeta is not clear; its dull within the unit disk, and its not quite obvious what the continuation is outside of the disk. XXX ToDo: get the full analytic structure.

12. CURIOSITIES

We list here some intriguing forms that suggest further relationships.

The Pochhammer symbol $(a)_n = \Gamma(a+n)/\Gamma(a)$ obeys a *dimidiation formula* that is reminiscent of the Bernoulli map:

$$\begin{aligned} (a)_{2n} &= 2^{2n} \left(\frac{a}{2}\right)_n \left(\frac{a+1}{2}\right)_n \\ (a)_{2n+1} &= 2^{2n+1} \left(\frac{a}{2}\right)_n \left(\frac{a+1}{2}\right)_n \end{aligned}$$

13. CONCLUSIONS

Apologies for the format of this paper.

APPENDIX A. APPENDIX A

This section needs to be re-written, merged, abolished. I think what its saying its mostly correct, but its misleading; there's a more enlightening, better treatment possible, by focusing on topologies.

A.1. Fourier Representation. Review of standard Fourier series techniques. In traditional notation, for some (periodic) function $f(x)$ one writes the Fourier Series as

$$(A.1) \quad f(x) = \sum_{n=-\infty}^{\infty} a_n \cos 2\pi n x + b_n \sin 2\pi n x$$

where the conjugates of f are given by

$$(A.2) \quad a_n = \int_0^1 f(x) \cos(2\pi n x) dx$$

and

$$(A.3) \quad b_n = \int_0^1 f(x) \sin(2\pi n x) dx$$

Moving over to bra-ket notation, we can define the Fourier-space basis vectors $|em\rangle$ in terms of their components in coordinate space. These components are $\langle x|em\rangle = \exp(i2\pi m x)$. The conjugate vectors $\langle en|$ have an equally simple representation: $\langle en|x\rangle = \exp(-i2\pi n x)$. One has the usual sense of orthogonality over coordinate space in that

$$(A.4) \quad \langle em|en\rangle = \int_0^1 dx \langle em|x\rangle \langle x|en\rangle = \int_0^1 dx \exp(2\pi i(n-m)x) = \delta_{nm}$$

and the traditional presentation of the Fourier Series is a statement of completeness over coordinate space, in that for an arbitrary square-integrable coordinate-space function $f(x) = \langle x|f\rangle$ one has

$$(A.5) \quad \begin{aligned} f(x) = \langle x|f\rangle &= \sum_{n=-\infty}^{\infty} \langle x|en\rangle \langle en|f\rangle \\ &= \sum_{n=-\infty}^{\infty} \exp(i2\pi n x) \int_0^1 dy \langle en|y\rangle \langle y|f\rangle \\ &= \sum_{n=-\infty}^{\infty} \exp(i2\pi n x) \int_0^1 dy \exp(-i2\pi n y) f(y) \\ &= \int_0^1 dy \delta(x-y) f(y) \end{aligned}$$

Thus, one is accustomed to the notion of having an identity operator of the form $1_F = \sum_{m=-\infty}^{\infty} |em\rangle \langle em|$ because it has the matrix elements that one expects in both the Fourier space and in coordinate space: that is, $\langle em|1_F|en\rangle = \delta_{nm}$ and $\langle x|1_F|y\rangle = \delta(x-y)$.

Thus, in light of this perfectly ordinary standard textbook behavior, the following shall be surprising. The matrix elements of this operator, expressed in the polynomial basis, are not only non-trivial, but are divergent. That is, one can be lulled into believing that $\langle m|1_F|n\rangle = \delta_{nm}$ for the polynomial basis, and indeed, by performing the operations in a certain order, one can certainly show this. However, reversing the order of operations shows that what might seem like simple operations can in fact be quite treacherous.

We begin by writing the components of the vector $|em\rangle$ in the polynomial-space representation:

$$\begin{aligned}
\langle n|em\rangle &= \int_0^1 dx \langle n|x\rangle \langle x|em\rangle \\
&= \int_0^1 dx \frac{(-)^n}{n!} \delta^{(n)}(x) e^{i2\pi mx} \\
&= \int_0^1 dx \frac{\delta(x)}{n!} \frac{d^n}{dx^n} e^{i2\pi mx} \\
&= \frac{(i2\pi m)^n}{n!}
\end{aligned}
\tag{A.6}$$

Essentially, this is nothing more than a plain-old Taylor's Series expansion of the exponential function. The conjugate vectors have a slightly trickier form. They are the Fourier components of monomials. For $m \neq 0$

$$\begin{aligned}
\langle em|n\rangle &= \int_0^1 dy \langle em|y\rangle \langle y|n\rangle \\
&= \int_0^1 \exp(-2\pi imy) y^n dy \\
&= \frac{-1}{2\pi im} + \frac{n}{2\pi im} \int_0^1 \exp(-2\pi imy) y^{n-1} dy \\
&= -\frac{1}{2\pi im} \sum_{k=0}^{n-1} \frac{n!}{(n-k)!} \left(\frac{1}{2\pi im} \right)^k
\end{aligned}
\tag{A.7}$$

and, for $m = 0$, $\langle e0|n\rangle = 1/(n+1)$. Let us now try to explicitly evaluate the matrix elements of the Fourier identity operator in the polynomial representation. That is, we attempt to write the matrix elements of $1_F = \sum_{m=-\infty}^{\infty} |em\rangle \langle em|$

$$\begin{aligned}
\langle p|1_F|n\rangle &= \sum_{m=-\infty}^{\infty} \langle p|em\rangle \langle em|n\rangle \\
&= \sum_{m=-\infty}^{\infty} \left[\delta_{p0} + (1 - \delta_{p0}) \frac{(2\pi im)^p}{p!} \right] \left[\frac{\delta_{m0}}{n+1} - \frac{(1 - \delta_{m0})}{2\pi im} \sum_{k=0}^{n-1} \frac{n!}{(n-k)!} \left(\frac{1}{2\pi im} \right)^k \right]
\end{aligned}
\tag{A.8}$$

We need only to look at the relatively simple matrix element $n = 1$, $p \neq 0$ to see the misery of this expression:

$$\langle p \neq 0|1_F|n = 1\rangle = \frac{(2\pi i)^p}{p!} \sum_{m=1}^{\infty} \frac{m^p}{2\pi im}
\tag{A.9}$$

One can try to rescue the situation by making the Ansatz that the summation should have been replaced by $\zeta(1-p)$ which is regular, but already this is dangerous. What is perhaps the more surprising is that one might have expected this kind of trouble from the polynomial completeness relationship $\mathbb{I}_A = \sum_{n=0}^{\infty} |n\rangle \langle n|$ because it ranges only over analytic functions: its essentially a statement of the idea that analytic functions are expressible through a series expansion in a variable. Functions that are not infinitely differentiable more-or-less lie in the kernel of \mathbb{I}_A . However, we'd expect 1_F to be more faithful, as it would seem to venture over square-integrable functions. Thus, such a simple failing is surprising.

The goal here is to simply present a signpost warning, as we make heavy use of these techniques in the sections that follow, where we work with functions that are differentiable-nowhere or worse.

A.2. The Koopman Operator. The Koopman operator is in a certain sense conjugate to the Frobenius-Perron operator, and defines how observables evolve. Given a density $\rho(x)$ we say that the observation of a function $f(x)$ by ρ is

$$(A.10) \quad \langle f \rangle_\rho = \int_0^1 f(x)\rho(x) dx$$

The term ‘‘observable’’ comes from usage in Quantum Mechanics, where $f(x)$ is associated with the eigenvalues of an operator. We do not need to appeal to these operator equations for the following development. The Koopman operator K gives the change in f when U acts on ρ , thus:

$$(A.11) \quad K_g : \langle f \rangle_\rho \rightarrow \langle K_g f \rangle_\rho = \int_0^1 [K_g f](x)\rho(x) dx = \int_0^1 f(x)[U_g \rho](x) dx$$

In Dirac bra-ket notation, we have

$$(A.12) \quad \begin{aligned} \int_0^1 f(x)[U_g \rho](x) dx &= \int_0^1 \langle x|U_g|\rho \rangle \langle x|f \rangle dx \\ &= \int_0^1 dx \int_0^1 dy \langle x|U_g|y \rangle \langle y|\rho \rangle \langle x|f \rangle dx \end{aligned}$$

and so we have

$$(A.13) \quad [K_g f](y) = \int_0^1 \langle x|U_g|y \rangle \langle x|f \rangle dx = \int_0^1 U_g(x,y)f(x) dx = \int_0^1 \delta(x - g(y))f(x) dx$$

This gives the action of the Koopman operator in a coordinate-space representation. As is the recurring theme, different representations can lead to different results. In the coordinate-space representation, the Koopman operator appears to be the transpose of the Frobenius-Perron operator, in that $K(x,y) = U(y,x)$. However, in a general representation, whether the Koopman operator is the transpose or the complex conjugate or something else needs to be determined on a case-by-case basis, with an appeal to the particular operator $g(x)$ and the representations on which it works.

A.3. Topologically Conjugate Maps. Conjugation of the function that generates the map will provide, in general, another map that behaves exactly the same as the first, as long as the conjugating function is a 1-1 and onto diffeomorphism. That is, if ϕ is invertible, so that

$$(A.14) \quad \gamma = \phi \circ g \circ \phi^{-1}$$

then γ will iterate the same way that g does: $\gamma^n = \phi \circ g^n \circ \phi^{-1}$. The orbit of any point x under the map g is completely isomorphic to the orbit of a point $y = \phi(x)$ under the map γ . Because the (chaotic) point dynamics of these two maps are isomorphic, we expect just about any related construction and analysis to show evidence of this isomorphism.

In particular, we expect that the Koopman and Frobenius-Perron operators for γ are conjugate to those for g :

$$(A.15) \quad U_\gamma = U_\phi^{-1} U_g U_\phi$$

XXX ToDo derive the above. Show that eigenvalues are preserved. The most trivial way to see that the eigenvalues are unchanged is through the formal definition of the characteristic polynomial for this operator, which is

$$(A.16) \quad p_U(\lambda) = \det[U_g - \lambda \mathbb{I}]$$

Just as in the finite-dimensional case, a similarity transform commutes inside the determinant, leaving the characteristic polynomial unchanged. XXX ToDo a more correct, non-formal proof that the eigenvalues are preserved.

Note that in the construction of this proof, we invoke the Jacobian $|d\phi(y)/dy|_{y=\phi^{-1}(x)}$ and thus, in order to preserve the polynomial-rep eigenvalues, the conjugating function must be a diffeomorphism; a homeomorphism does not suffice. We will show an example below of a conjugating function that is highly singular, and thus the Jacobian does not exist (in the ordinary sense). When the conjugating function is sufficiently singular, then U_ϕ cannot be coherently defined. As a result, one can have conjugate maps with completely isomorphic point dynamics, but the eigenvalue spectra associated with these maps will *not* be identical.

A.4. The Topological Zeta. Another interesting quantity is the topological zeta function associated with the transfer operator. It is formally defined by

$$(A.17) \quad \zeta_{U_g}(t) = \frac{1}{\det[\mathbb{I} - tU_g]}$$

and embeds number-theoretic information about the map. Using standard formal manipulations on operators, one can re-write the above as the operator equation

$$(A.18) \quad \zeta_{U_g}(t) = \exp \sum_{k=1}^{\infty} \frac{t^k}{k} \text{Tr} U_g^k$$

Of associated interest is the Maclaurin Series

$$(A.19) \quad t \frac{d}{dt} \log \zeta_{U_g}(t) = \sum_{k=1}^{\infty} n_k t^k$$

where we can read off $n_k = \text{Tr} U_g^k$. From graph theory and the theory of dynamical systems, it is known that the n_k correspond to the number of periodic orbits of length k . In the context of dynamical systems, this zeta is often referred to as the Artin-Mazur Zeta function. In the context of graph theory, it is referred to as the Ihara Zeta. Both are connected to the Selberg Zeta.

The standard definition of the Ihara Zeta applies only to the adjacency matrix of finite-sized graphs. Adjacency matrices only have (non-negative) integer entries as matrix elements. Thus, we ask: given an appropriate basis, can an infinite-dimensional transfer operator be written so as to have integer entries as matrix elements?

The standard definition of the Artin-Mazur Zeta function requires that the number of fixed points (periodic orbits) be a finite number. For the operators that we are studying, there will in general be (countably) infinite number of periodic orbits. Yet the zeta will still be well defined, although the coefficients of the Maclaurin expansion will not be integers. Can these be reinterpreted as a density or measure?

APPENDIX B. JONQUIÈRE'S IDENTITY

This appendix establishes the relationship between the functions $\beta(x, s)$ and the Hurwitz zeta function

$$(B.1) \quad \zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}$$

This relationship is well-known; for example, it is reviewed in the Bateman manuscripts in the guise of an identity on the Lerch transcendental[?]; see also Wikipedia on Polylogarithms[?]. As a relation on polylogarithms, it is known as Jonquière's identity[?, Section 7.12.2][?].

One may confirm this by following a very old-fashioned recipe for obtaining the functional relation for a zeta-like sum. Start by expressing the gamma function as

$$(B.2) \quad \int_0^\infty dy e^{-2\pi ny} y^{s-1} = \frac{\Gamma(s)}{(2\pi n)^s}$$

Substituting into the expression for β and performing the sum, one may write

$$(B.3) \quad \beta(x; s) = 2s \int_0^\infty dy \frac{y^{s-1}}{\exp(-2\pi i(x+iy)) - 1}$$

Then, following a traditional trick [?, pp 13 ff], re-write this as a contour integral

$$(B.4) \quad \beta(x; s) = \frac{-is}{\sin \pi s} \oint \frac{(-y)^s}{\exp(-2\pi i(x+iy)) - 1} \frac{dy}{y}$$

where the contour is taken to extend from $+\infty + i\epsilon$, running just above the positive real axis, to the origin, circling the origin in a clockwise fashion, and returning to $+\infty - i\epsilon$ just under the real axis. The contour essentially encloses the cut of the logarithm in the expression $(-y)^s = \exp s \log(-y)$. The old fashioned recipe calls for closing the contour at infinity (in a counter-clockwise direction) and then taking the dubious step of asserting Cauchy's Theorem to equate the integral around the cut to the sum of the poles, where we note that we have a pole whenever $x + iy = n$ for some integer n . By doing this we get the formal summation

$$(B.5) \quad \frac{i \sin \pi s}{s} \beta(x; s) = \exp\left(\frac{i\pi s}{2}\right) \sum_{n=-\infty}^{\infty} (n-x)^{s-1}$$

This is a "formal sum", since the preceding steps required taking $\Re s > 1$ whereas now one needs to take $\Re s < 0$. This is a bit of jiggery-pokery that is common for this type of presentation; a different set of tools is required to do better. So we proceed, ignoring these difficulties. Re-write this sum as

$$(B.6) \quad \frac{i \sin \pi s}{s} \beta(x; s) = \exp\left(\frac{i\pi s}{2}\right) \left[\sum_{n=0}^{\infty} (n+(1-x))^{s-1} + e^{-i\pi(s-1)} \sum_{n=0}^{\infty} (n+x)^{s-1} \right]$$

where we were mindful to rotate counter-clockwise for $n < 0$ when replacing $(-)^n$ by $e^{-i\pi n}$ instead of the sloppy and incorrect $e^{i\pi n}$. Recognizing the sums as the Hurwitz Zeta, this then gives the desired result:

$$(B.7) \quad \beta(x; s) = \frac{is}{\sin \pi s} \left[e^{-i\pi s/2} \zeta(1-s, x) - e^{i\pi s/2} \zeta(1-s, 1-x) \right]$$

It is straightforward to invert this and solve for ζ ; one gets

$$(B.8) \quad \zeta(1-s, x) = \frac{1}{2s} \left[e^{-i\pi s/2} \beta(x; s) + e^{i\pi s/2} \beta(1-x; s) \right]$$

thus proving the assertion that the Hurwitz zeta is an eigenfunction of the Bernoulli operator, with eigenvalue 2^{s-1} . To verify the correctness of the above steps, expand the exponentials in terms of their real and imaginary parts, to find that

$$(B.9) \quad \zeta(z, x) = \frac{2\Gamma(1-z)}{(2\pi)^{1-z}} \left[\sin\left(\frac{\pi z}{2}\right) \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n^{1-z}} + \cos\left(\frac{\pi z}{2}\right) \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n^{1-z}} \right]$$

which agrees with standard textbook presentations of the Hurwitz zeta; see for example [?, Thm 12.6, Ex 12.2].

APPENDIX C. VISUALIZING THE ZETA FUNCTION

This section provides a visualization and a simple discussion of the analytic properties of the zeta eigenfunctions. The main point made is that the eigenfunctions are smooth, that is, infinitely differentiable (C^∞) in x for all x except at the endpoints $x = 0, 1$, where there is an essential singularity. There are eigenfunctions that have eigenvalues greater than one; these, while quite smooth, are not square-integrable: they are divergent at $x = 0, 1$. However, in all other respects, the eigenfunctions are analytically well-behaved, even if a bit “lumpy” and uneven, as the following graphs show.

There are eigenfunctions with eigenvalues greater than one, essentially because the Hurwitz zeta can be analytically continued to everywhere on the complex plane except for a simple pole at $z = 1$. Examining these eigenfunctions, one quickly discovers that these are not square-integrable: they have singularities located at $x = 0, 1$. That is, for $\Re z > 0$, the Hurwitz zeta $\zeta(z, x)$ has a clear singularity x^{-z} at $x = 0$. Separating this out explicitly, one has

$$(C.1) \quad \frac{\sin \pi s}{is} \beta(x; s) = \frac{e^{-i\pi s/2}}{x^{1-s}} - \frac{e^{i\pi s/2}}{(1-x)^{1-s}} + e^{-i\pi s/2} (\zeta(1-s, 1+x)) - e^{i\pi s/2} (\zeta(1-s, 2-x))$$

The first part of the equation above encapsulates the singularities at $x = 0, 1$ that occur when working with eigenvalues $|\lambda| = |2^{-s}| > 1$, that is, with $\Re s < 0$. The remaining term is well-behaved and is shown in figure C.3.

When $|\lambda| = |2^{-s}| < 1/2$, that is, when $\Re s > 1$, there is no singularity, and $\beta(x; s)$ is finite on the entire interval $x \in [0, 1]$, including the endpoints. For $1/2 < \Re s \leq 1$ there is a bit of funny-business at the endpoints, that is, there is a weak divergence there, but the function overall remains square-integrable. Things break loose after that, with the exception of $s = 0$, where we have $\beta(x; 0) = -1$, a constant independent of x . This essentially follows from the nature of differentiation on the Bernoulli polynomials, which we’ll see below. For s near zero, the function $\beta(x; s)$ has severe ringing artifacts in x , suffering from a variation of Gibbs phenomenon.

The function $\beta(x; s)$ is C^∞ for $x \in (0, 1)$ but not at the endpoints $x = 0, 1$. This can be easily seen by writing the derivative

$$(C.2) \quad \frac{d}{dx} \beta(x; s) = 2\pi i \beta(x; s-1)$$

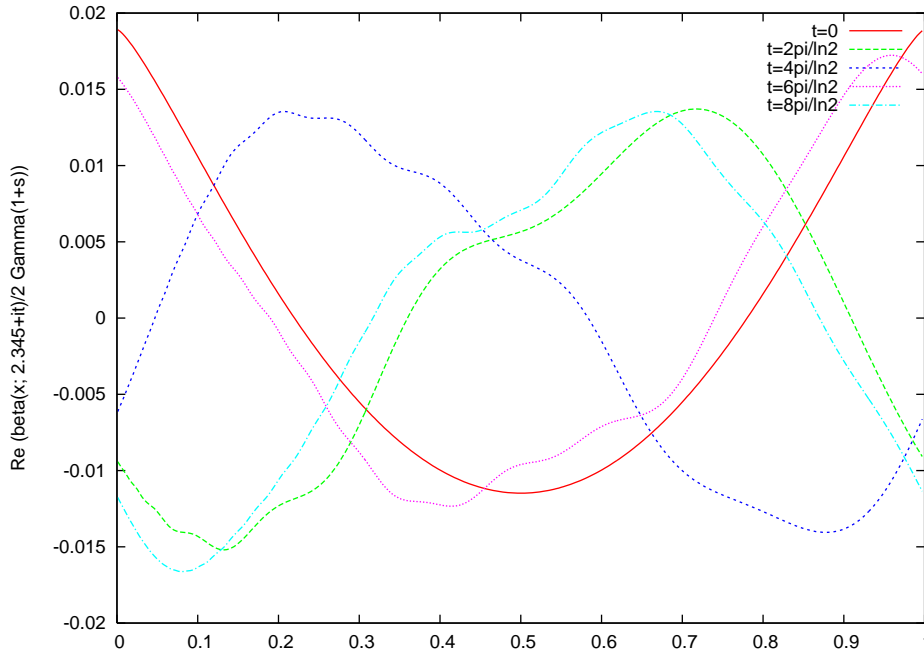
and so even if one starts with $\Re s > 1$, each derivative moves one step closer to the danger zone.

APPENDIX D. APPENDIX: TO-DO LIST

In order to give a proper and complete treatment of the subject, the topics listed below should be presented/reviewed/understood.

- The Bernoulli process in probability theory is one of the simplest Markov processes. Understood as a Markov process, it has a number of generalizations.
- Discuss entropy.
- Discuss the Riesz representation theorem.
- Discuss symbolic dynamics in two letters. Mention one-dimensional tiling, mention Fibonacci tiling. Mention Lindenmeyer systems. Mention subshifts of finite type. Mention how subshifts are solved.
- Discuss connection to the Cantor set. viz, all strings in two letters.

FIGURE C.1. Real part of Beta



This figure illustrates a family of eigenvectors $\beta(x; s)$ having the same eigenvalue. To be precise, the illustration shows

$$\Re \left(\frac{\beta(x; s)}{2\Gamma(1+s)} \right) = \Re \left((\pi)^{-s} \text{Li}_s(e^{2\pi i x}) \right)$$

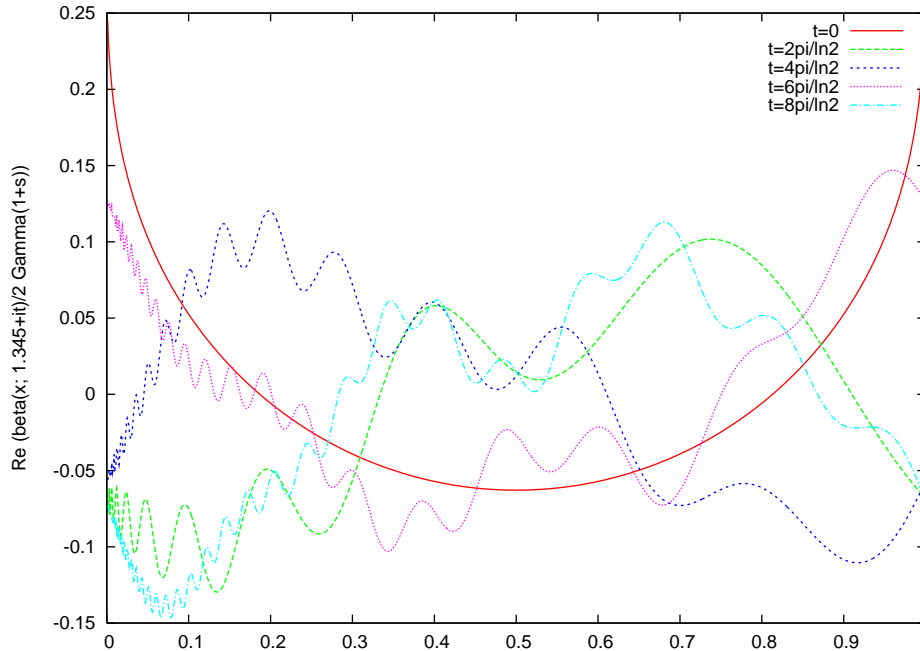
for values of $s = \sigma + 2\pi i n / \ln 2$ for $\sigma = 2.345$ and $n = 0, 1, 2, 3, 4$. Note that $\Gamma(1+s)$ gets very small as n gets large, and so the normalization brings them to visually comparable values.

Although these curves clearly look very lumpy, they are C^∞ for all $x \in (0, 1]$ but not at the endpoint $x = 0$. At the endpoint, the derivative becomes divergent after just a few derivatives, where the curves are behaving essentially as x^{s-1} . This pending divergence and the true nature of the lumpiness is made clear in the next graphic.

Although these curves appear to be sine-wave-like, it is perhaps more correct to think of them as being Bernoulli-polynomial-like. That is, the general case will look similar to the polynomial $B_{[\Re s]}(x)$. Of course, $B_k(x)$ for $k \geq 3$ is very sine-wave like, so the general wave-like nature holds for most values of s . The first curve shown above, for $n = 0$, generally resembles $B_2(x)$, which is a parabola.

- Discuss free groups, discuss Cayley tree, discuss group presentation, and connect it up to this.
- Discuss ergodicity, the ergodic theorem, and how its related to this. Discuss how the Bernoulli map is a “bad” example of ergodicity. Hypothesis: are all transfer operators that are triangular (have a polynomial basis) equivalent to “bad” ergodic sequences? Or are there good (uniformly converging) ergodic sequences that can result from triangular transfer operators?

FIGURE C.2. Real part of Beta



This figure illustrates a family of eigenvectors $\beta(x; s)$ having the same eigenvalue. To be precise, the illustration shows

$$\Re\left(\frac{\beta(x; s)}{2\Gamma(1+s)}\right) = \Re\left((2\pi)^{-s} \text{Li}_s(e^{2\pi i x})\right)$$

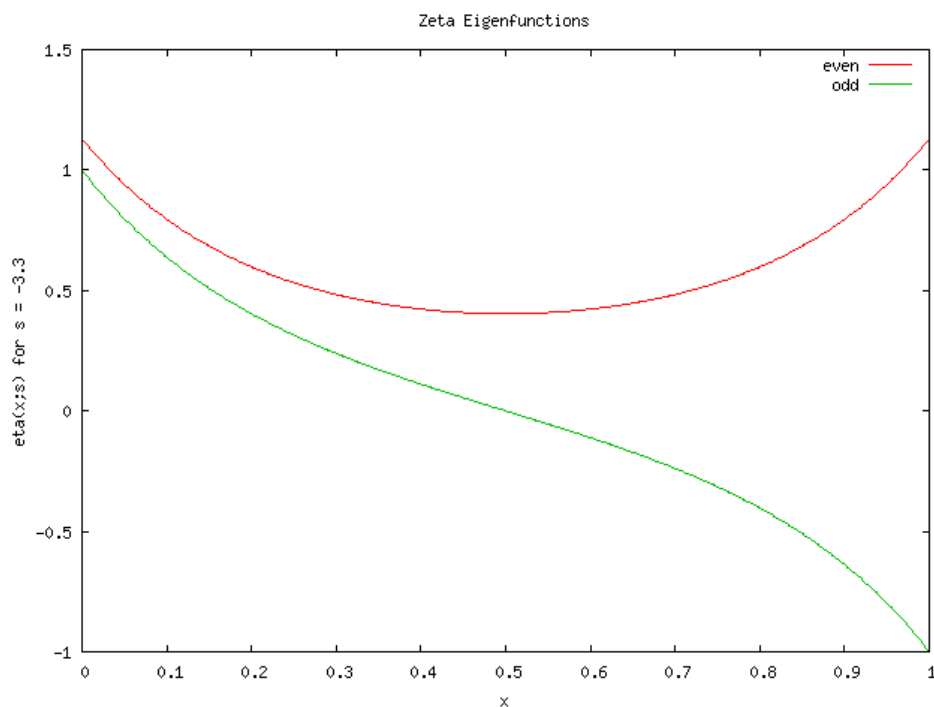
for values of $s = \sigma + 2\pi i n / \ln 2$ for $\sigma = 1.345$ and $n = 0, 1, 2, 3, 4$. The contribution of $\Re x^{\sigma-1} = \cos((\sigma-1) \ln x)$ is more clearly visible in this image; at smaller amplitudes, this ringing makes the curves look lumpy.

- Discuss wavelet transforms, point out their dyadic nature, point out their relationship to this mess; point out how wavelet transforms are a good tool for working with this kind of dyadic data.
- While this paper focuses primarily on the polynomial and square-integrable eigenfunctions of the Bernoulli operator, there is also a class of non-differentiable, non-integrable eigenfunctions that can be precisely defined in terms of the shift operator on a one-dimensional lattice model, the Ising model of statistical mechanics. The Ising model has a natural topology that is distinctly different from the topology of the real number line; this alternate topology allows for a much, much larger set of eigenfunctions. This is discussed in greater detail in [?].

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FIGURE C.3. The Non-Singular Part of the Divergent Eigenfunctions



This figure shows

$$\eta_{\text{even}}(x; \sigma) = \frac{\cos \pi \sigma / 2}{\sigma} [\beta(x; \sigma) + \beta(1-x; \sigma)] - x^{\sigma-1} - (1-x)^{\sigma-1}$$

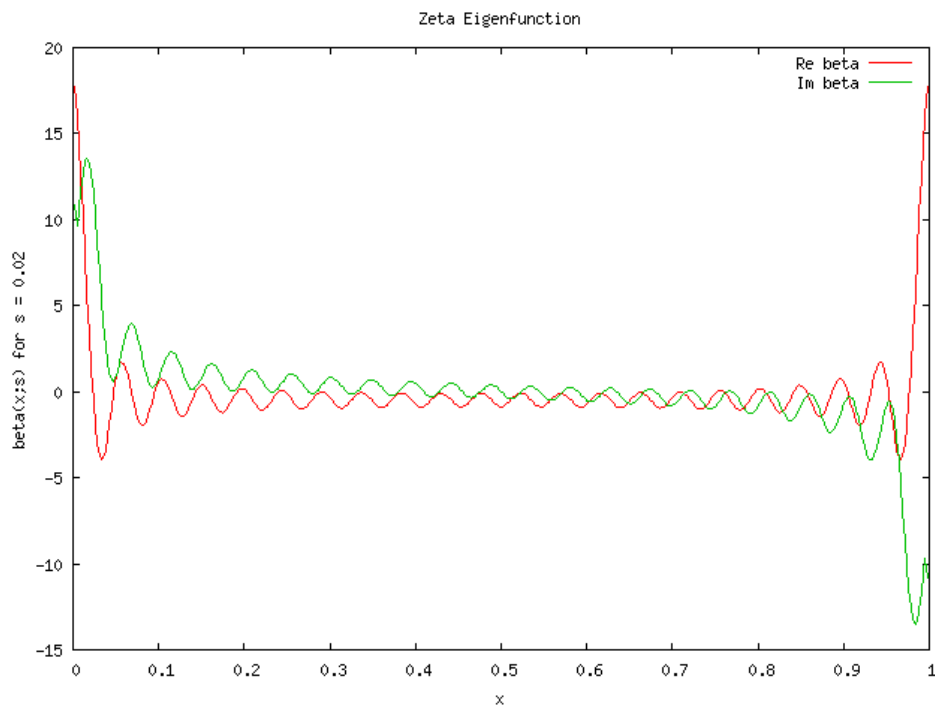
and

$$\eta_{\text{odd}}(x; \sigma) = \frac{\sin \pi \sigma / 2}{\sigma} [\beta(x; \sigma) - \beta(1-x; \sigma)] - x^{\sigma-1} + (1-x)^{\sigma-1}$$

for a value of $\sigma = -3.3$, corresponding to an eigenvalue of $9.85 = 2^{3.3}$. Except for the singularity, we see that the finite part of these eigenfunctions is very well behaved.

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FIGURE C.4. Ringing



This figure shows ringing/Gibbs phenomenon as s approaches zero. In the limit of $s = 0$, we expect the real part of β to approach the trivial eigenfunction $\lim_{s \rightarrow 0^+} \Re \beta(x; s) = -B_0(x) = -1$. As this graph shows, the function is indeed trying very desperately to get flat, with not much success. The ringing occurs only at $s = 0$; there is no problem with convergence near larger integers, where $\lim_{s \rightarrow n} \Re(-i)^s \beta(x; s) = -B_n(x)$ converges very smoothly and cleanly.

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